Instructions: As in previous homeworks. See Old HW 11 for tips and examples on how to write NP-completeness proofs.

Problem 11.1: Given an undirected graph $G$, we want to decide whether $G$ contains a spanning tree where every node has degree at most 4. Prove that this problem is NP-complete.

[Hint: you may assume that the HAMILTONIAN PATH problem is NP-complete.]

Problem 11.2: We need to schedule final exams for $N$ classes. We want to minimize the number of days, but don’t want any students to take more than 2 exams on a single day.

One way to formulate this problem is as follows: There are $M$ students, and for each $j = 1, \ldots, M$, we are given a set $S_j \subseteq \{1, \ldots, N\}$ of the classes that student $j$ is taking. We are also given an integer $D$. We want to decide whether there exists a function $f : \{1, \ldots, N\} \rightarrow \{1, \ldots, D\}$ such that for every $j$ and $k$, the number of elements in $\{x \in S_j \mid f(x) = k\}$ is at most 2.

Prove that this problem is NP-complete.

[Hint: you may assume that 3-COLORING is NP-complete. Observe that if we create 4 copies of a vertex $v$ and $D = 3$, then two copies of $v$ must have the same $f$ value. For each edge $uv$, create a constant number of sets (of size 3 or 4)…]

Problem 11.3: Consider the following version of the CROSSWORD-PUZZLE problem:

Input: $A_1, \ldots, A_m, B_1, \ldots, B_n$, where each $A_i$ is a finite set of length-$n$ strings and each $B_j$ is a finite set of length-$m$ strings, over a finite alphabet $\Sigma$.

Output: “yes” iff there exists an $m \times n$ table $T$ of symbols such that for each $i = 1, \ldots, m$, the $i$-th row of $T$ is a string in the set $A_i$, and for each $j = 1, \ldots, n$, the $j$-th column of $T$ is a string in the set $B_j$.

Example: on the input $A_1 = \{\text{CAT, DOG}\}$, $A_2 = \{\text{CAT, APE, AGO}\}$, $A_3 = \{\text{BAD, BEE}\}$, $B_1 = \{\text{CAE, DAB}\}$, $B_2 = \{\text{APE, EGG}\}$, and $B_3 = \{\text{GOD, TEE}\}$, the answer is yes, with the following solution $T$:

CAT
APE
BEE

Prove that CROSSWORD-PUZZLE is NP-complete.

[Hint: reduce from 3SAT. Given a 3CNF formula $F$ with variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$, let $b_j$ be the length-$m$ binary string (over $\Sigma = \{0, 1\}$) such that the $i$-th bit is 1 iff $x_j$ appears in $C_i$, and let $b'_j$ be the length-$m$ binary string such that the $i$-th bit is 1 iff $\overline{x_j}$ appears in $C_i$. Define $B_j = \{b_j, b'_j\}$, which contains 2 strings. Now define $A_i$ to contain 7 appropriately chosen strings…]