Breadth First Search, Dijkstra’s Algorithm for Shortest Paths

Lecture 17
Tuesday, March 19, 2019
Part I

Breadth First Search
Overview

(A) **BFS** is obtained from **BasicSearch** by processing edges using a queue data structure.

(B) It processes the vertices in the graph in the order of their shortest distance from the vertex \( s \) (the start vertex).

As such...

1. **DFS** good for exploring graph structure
2. **BFS** good for exploring *distances*
xkcd take on DFS

[Comic panels show a person preparing for a date, considering various emergencies, and ending up in a discussion about snakes.

1. What situations might I prepare for?
   1) Medical emergency
   2) Dancing
   3) Food too expensive

2. Okay, what kinds of emergencies can happen?
   1) Snake bite
   2) Lightning strike
   3) Fall from chair

3. Hmm, which snakes are dangerous? Let's see...
   1) Corn snake
   2) Garter snake
   3) Copperhead

The research comparing snake venoms is scattered and inconsistent. I'll make a spreadsheet to organize it.

I'm here to pick you up. You're not dressed?

By the way, the Inland Taipan has the deadliest venom of any snake.

I really need to stop using depth-first searches.
### Queue Data Structure

A **queue** is a list of elements which supports the operations:

1. **enqueue**: Adds an element to the end of the list
2. **dequeue**: Removes an element from the front of the list

Elements are extracted in **first-in first-out (FIFO)** order, i.e., elements are picked in the order in which they were inserted.
BFS Algorithm

Given (undirected or directed) graph \( G = (V, E) \) and node \( s \in V \)

**BFS**

Mark all vertices as unvisited
Initialize search tree \( T \) to be empty
Mark vertex \( s \) as visited
set \( Q \) to be the empty queue
\( \texttt{enqueue}(Q, s) \)

while \( Q \) is nonempty do
  \( u = \texttt{dequeue}(Q) \)
  for each vertex \( v \in \text{Adj}(u) \)
    if \( v \) is not visited then
      add edge \((u, v)\) to \( T \)
      Mark \( v \) as visited and \( \texttt{enqueue}(v) \)

**Proposition**

\( \texttt{BFS}(s) \) runs in \( O(n + m) \) time.
BFS: An Example in Undirected Graphs

1. [1]
BFS: An Example in Undirected Graphs

1. [1]
2. [2,3]
BFS: An Example in Undirected Graphs

1. [1]
2. [2,3]
3. [3,4,5]
BFS: An Example in Undirected Graphs

1. $[1]$
2. $[2,3]$
3. $[3,4,5]$
4. $[4,5,7,8]$

The BFS tree is the set of purple edges.
BFS: An Example in Undirected Graphs

1. [1]
2. [2,3]
3. [3,4,5]

4. [4,5,7,8]
5. [5,7,8]

BFS tree is the set of purple edges.

Chan, Har-Peled, Hassanieh (UIUC)
BFS: An Example in Undirected Graphs

1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]

Chan, Har-Peled, Hassanieh (UIUC)
BFS: An Example in Undirected Graphs

1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
BFS: An Example in Undirected Graphs

1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
BFS: An Example in Undirected Graphs

1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
9. []
BFS: An Example in Undirected Graphs

1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
9. []

BFS tree is the set of purple edges.
BFS: An Example in Undirected Graphs

1. [1]  
2. [2,3]  
3. [3,4,5]  
4. [4,5,7,8]  
5. [5,7,8]  
6. [7,8,6]  
7. [8,6]  
8. [6]  
9. []

**BFS** tree is the set of purple edges.
BFS: An Example in Undirected Graphs

1. [1]
2. [2,3]
3. [3,4,5]
4. [4,5,7,8]
5. [5,7,8]
6. [7,8,6]
7. [8,6]
8. [6]
9. []

BFS tree is the set of purple edges.
BFS: An Example in Directed Graphs

Definition
A directed graph (also called a digraph) is $G = (V, E)$, where $V$ is a set of vertices or nodes and $E \subseteq V \times V$ is a set of ordered pairs of vertices called edges.
BFS with Distance

**BFS(s)**
Mark all vertices as unvisited; for each \( v \) set \( \text{dist}(v) = \infty \)
Initialize search tree \( T \) to be empty
Mark vertex \( s \) as visited and set \( \text{dist}(s) = 0 \)
set \( Q \) to be the empty queue

**enqueue(s)**

while \( Q \) is nonempty do
  \( u = \text{dequeue}(Q) \)
  for each vertex \( v \in \text{Adj}(u) \) do
    if \( v \) is not visited do
      add edge \((u, v)\) to \( T \)
      Mark \( v \) as visited, enqueue(\( v \))
      and set \( \text{dist}(v) = \text{dist}(u) + 1 \)
Properties of BFS: Undirected Graphs

Theorem

The following properties hold upon termination of $\text{BFS}(s)$

(A) The search tree contains exactly the set of vertices in the connected component of $s$.

(B) If $\text{dist}(u) < \text{dist}(v)$ then $u$ is visited before $v$.

(C) For every vertex $u$, $\text{dist}(u)$ is the length of a shortest path (in terms of number of edges) from $s$ to $u$.

(D) If $u, v$ are in connected component of $s$ and $e = \{u, v\}$ is an edge of $G$, then $|\text{dist}(u) - \text{dist}(v)| \leq 1$.

$$\text{dist}(s) = d$$

$$\text{dist}(u) = d$$

$$\text{dist}(v) = d + 3$$
Properties of BFS: Directed Graphs

**Theorem**

The following properties hold upon termination of \( \text{BFS}(s) \):

(A) The search tree contains exactly the set of vertices reachable from \( s \)

(B) If \( \text{dist}(u) < \text{dist}(v) \) then \( u \) is visited before \( v \)

(C) For every vertex \( u \), \( \text{dist}(u) \) is indeed the length of shortest path from \( s \) to \( u \)

(D) If \( u \) is reachable from \( s \) and \( e = (u, v) \) is an edge of \( G \), then
\[
\text{dist}(v) - \text{dist}(u) \leq 1.
\]

*Not necessarily the case that \( \text{dist}(u) - \text{dist}(v) \leq 1. \)*
**BFS with Layers**

**BFS\text{Layers}(s):**

Mark all vertices as unvisited and initialize $T$ to be empty.
Mark $s$ as visited and set $L_0 = \{s\}$

$i = 0$

while $L_i$ is not empty do

initialize $L_{i+1}$ to be an empty list

for each $u$ in $L_i$ do

for each edge $(u, v) \in \text{Adj}(u)$ do

if $v$ is not visited

mark $v$ as visited

add $(u, v)$ to tree $T$

add $v$ to $L_{i+1}$

$i = i + 1$

Running time: $O(n + m)$
**BFS with Layers**

**BFSLayers**($s$):

Mark all vertices as unvisited and initialize $T$ to be empty
Mark $s$ as visited and set $L_0 = \{s\}$

$i = 0$

**while** $L_i$ is not empty **do**

initialize $L_{i+1}$ to be an empty list

**for** each $u$ in $L_i$ **do**

**for** each edge $(u,v) \in \text{Adj}(u)$ **do**

if $v$ is not visited

mark $v$ as visited

add $(u,v)$ to tree $T$

add $v$ to $L_{i+1}$

$i = i + 1$

**Running time:** $O(n + m)$
Example
BFS with Layers: Properties

**Proposition**

The following properties hold on termination of \texttt{BFSLayers}(s).

1. \texttt{BFS}\texttt{Layers}(s) outputs a \texttt{BFS} tree
2. $L_i$ is the set of vertices at distance exactly $i$ from $s$
3. If $G$ is undirected, each edge $e = \{u, v\}$ is one of three types:
   
   1. \textbf{tree} edge between two consecutive layers
   2. non-tree \textbf{forward/backward} edge between two consecutive layers
   3. non-tree \textbf{cross-edge} with both $u, v$ in same layer
   4. $\Rightarrow$ Every edge in the graph is either between two vertices that are either (i) in the same layer, or (ii) in two consecutive layers.
Example

A directed graph (also called a digraph) is $G = (V, E)$, where $V$ is a set of vertices or nodes and $E \subseteq V \times V$ is set of ordered pairs of vertices called edges.

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BFS with Layers: Properties

For directed graphs

Proposition

The following properties hold on termination of BFS\(_\text{Layers}(s)\), if \(G\) is directed.

For each edge \(e = (u, v)\) is one of four types:

1. a tree edge between consecutive layers, \(u \in L_i, v \in L_{i+1}\) for some \(i \geq 0\)
2. a non-tree forward edge between consecutive layers
3. a non-tree backward edge
4. a cross-edge with both \(u, v\) in same layer
Part II

Shortest Paths and Dijkstra’s Algorithm
### Shortest Path Problems

**Input** A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

1. Given nodes $s, t$ find shortest path from $s$ to $t$.
2. Given node $s$ find shortest path from $s$ to all other nodes.
3. Find shortest paths for all pairs of nodes.
Shortest Path Problems

Input: A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$,
\[ \ell(e) = \ell(u, v) \] is its length.

1. Given nodes $s, t$ find shortest path from $s$ to $t$.
2. Given node $s$ find shortest path from $s$ to all other nodes.
3. Find shortest paths for all pairs of nodes.

Many applications!
Single-Source Shortest Paths:
Non-Negative Edge Lengths

1. Single-Source Shortest Path Problems
   1. **Input:** A (undirected or directed) graph $G = (V, E)$ with non-negative edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.
   2. Given nodes $s, t$ find shortest path from $s$ to $t$.
   3. Given node $s$ find shortest path from $s$ to all other nodes.
Single-Source Shortest Paths:
Non-Negative Edge Lengths

Single-Source Shortest Path Problems

1. **Input**: A (undirected or directed) graph $G = (V, E)$ with non-negative edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.
2. Given nodes $s, t$ find shortest path from $s$ to $t$.
3. Given node $s$ find shortest path from $s$ to all other nodes.

Restrict attention to directed graphs

Undirected graph problem can be reduced to directed graph problem - how?
Single-Source Shortest Path Problems

1. **Input**: A (undirected or directed) graph \( G = (V, E) \) with non-negative edge lengths. For edge \( e = (u, v) \), \( \ell(e) = \ell(u, v) \) is its length.

2. Given nodes \( s, t \) find shortest path from \( s \) to \( t \).

3. Given node \( s \) find shortest path from \( s \) to all other nodes.

Restrict attention to directed graphs

Undirected graph problem can be reduced to directed graph problem - how?

1. Given undirected graph \( G \), create a new directed graph \( G' \) by replacing each edge \( \{u, v\} \) in \( G \) by \( (u, v) \) and \( (v, u) \) in \( G' \).
2. set \( \ell(u, v) = \ell(v, u) = \ell(\{u, v\}) \)
3. Exercise: show reduction works. Relies on non-negativity!
1 Special case: All edge lengths are 1.
1 **Special case:** All edge lengths are 1.
   1 Run \textbf{BFS}(s) to get shortest path distances from s to all other nodes.
   2 \textbf{O}(m + n) time algorithm.
Special case: All edge lengths are 1.

- Run \textbf{BFS}(s) to get shortest path distances from s to all other nodes.
- \(O(m + n)\) time algorithm.

Special case: Suppose \(\ell(e)\) is an integer for all \(e\)? Can we use \textbf{BFS}?

\[ u \rightarrow v \]
\[ u \rightarrow d_1 \rightarrow d_2 \rightarrow v \]
Special case: All edge lengths are 1.

Run BFS(s) to get shortest path distances from s to all other nodes.

O(m + n) time algorithm.

Special case: Suppose ℓ(e) is an integer for all e? Can we use BFS? Reduce to unit edge-length problem by placing ℓ(e) − 1 dummy nodes on e.
Single-Source Shortest Paths via BFS

1 Special case: All edge lengths are 1.
   1 Run BFS(s) to get shortest path distances from s to all other nodes.
   2 \( O(m + n) \) time algorithm.

2 Special case: Suppose \( \ell(e) \) is an integer for all \( e \)? Can we use BFS? Reduce to unit edge-length problem by placing \( \ell(e) - 1 \) dummy nodes on \( e \).

3 Let \( L = \max_e \ell(e) \). New graph has \( O(mL) \) edges and \( O(mL + n) \) nodes. BFS takes \( O(mL + n) \) time. Not efficient if \( L \) is large.
Towards an algorithm

Why does **BFS** work?

**Lemma**
Let $G$ be a directed graph with non-negative edge lengths. Let $\text{dist}(s, v)$ denote the shortest path length from $s$ to $v$.

If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ shortest path from $s$ to $v_k$ then for $1 \leq i < k$:

$$s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$$

is shortest path from $s$ to $v_i$.

$\text{dist}(s, v_i) \leq \text{dist}(s, v_k)$. Relies on non-neg edge lengths.

**Proof.**
Suppose not. Then for some $i < k$ there is a path $P'$ from $s$ to $v_i$ of length strictly less than that of $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$. Then $P'$ concatenated with $v_i \rightarrow v_{i+1} \rightarrow \ldots \rightarrow v_k$ contains a strictly shorter path to $v_k$ than $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$. For the second part, observe that edge lengths are non-negative.
Towards an algorithm

Why does **BFS** work?

**BFS**($s$) explores nodes in increasing distance from $s$.
Towards an algorithm

Why does **BFS** work?

**BFS**(s) explores nodes in increasing distance from **s**

**Lemma**

Let **G** be a directed graph with non-negative edge lengths. Let \( \text{dist}(s, v) \) denote the shortest path length from **s** to **v**.

If \( s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k \) shortest path from **s** to **v_k** then for \( 1 \leq i < k \):

1. **s** = \( v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i \) is shortest path from **s** to **v_i**
2. \( \text{dist}(s, v_i) \leq \text{dist}(s, v_k) \). Relies on non-neg edge lengths.
Lemma

Let $G$ be a directed graph with non-negative edge lengths. Let $\text{dist}(s, v)$ denote the shortest path length from $s$ to $v$.

If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ shortest path from $s$ to $v_k$ then for $1 \leq i < k$:

1. $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is shortest path from $s$ to $v_i$
2. $\text{dist}(s, v_i) \leq \text{dist}(s, v_k)$. Relies on non-neg edge lengths.

Proof.

Suppose not. Then for some $i < k$ there is a path $P'$ from $s$ to $v_i$ of length strictly less than that of $s = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_i$. Then $P'$ concatenated with $v_i \rightarrow v_{i+1} \ldots \rightarrow v_k$ contains a strictly shorter path to $v_k$ than $s = v_0 \rightarrow v_1 \ldots \rightarrow v_k$. For the second part, observe that edge lengths are non-negative.
A proof by picture

Shortest path from $v_0$ to $v_6$
A proof by picture

Shorter path from $v_0$ to $v_4$

$s = v_0$

Shortest path from $v_0$ to $v_6$
A proof by picture

\[ s = v_0 \]

\[ v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \rightarrow v_5 \rightarrow v_6 \]

Shortest path from \( v_0 \) to \( v_6 \)

A shorter path from \( v_0 \) to \( v_6 \). A contradiction.
A Basic Strategy

Explore vertices in increasing order of distance from $s$: (For simplicity assume that nodes are at different distances from $s$ and that no edge has zero length)

Initialize for each node $v$, $\text{dist}(s, v) = \infty$
Initialize $X = \{s\}$,
\[
\text{for } i = 2 \text{ to } |V| \text{ do }
\]
(* Invariant: $X$ contains the $i-1$ closest nodes to $s$ *)
Among nodes in $V - X$, find the node $v$ that is the $i$'th closest to $s$
Update $\text{dist}(s, v)$
$X = X \cup \{v\}$
A Basic Strategy

Explore vertices in increasing order of distance from $s$:
(For simplicity assume that nodes are at different distances from $s$ and that no edge has zero length)

Initialize for each node $v$, $\text{dist}(s, v) = \infty$
Initialize $X = \{s\}$,
for $i = 2$ to $|V|$ do
  (* Invariant: $X$ contains the $i−1$ closest nodes to $s$ *)
  Among nodes in $V − X$, find the node $v$ that is the $i$’th closest to $s$
  Update $\text{dist}(s, v)$
  $X = X \cup \{v\}$

How can we implement the step in the for loop?
Finding the \( i \)th closest node

1. \( X \) contains the \( i - 1 \) closest nodes to \( s \)
2. Want to find the \( i \)th closest node from \( V - X \).

What do we know about the \( i \)th closest node?
Finding the $i$th closest node

1. $X$ contains the $i - 1$ closest nodes to $s$
2. Want to find the $i$th closest node from $V - X$.

What do we know about the $i$th closest node?

**Claim**

Let $P$ be a shortest path from $s$ to $v$ where $v$ is the $i$th closest node. Then, all intermediate nodes in $P$ belong to $X$. 
Finding the $i$th closest node

1. $X$ contains the $i - 1$ closest nodes to $s$
2. Want to find the $i$th closest node from $V - X$.

What do we know about the $i$th closest node?

**Claim**

Let $P$ be a shortest path from $s$ to $v$ where $v$ is the $i$th closest node. Then, all intermediate nodes in $P$ belong to $X$.

**Proof.**

If $P$ had an intermediate node $u$ not in $X$ then $u$ will be closer to $s$ than $v$. Implies $v$ is not the $i$'th closest node to $s$ - recall that $X$ already has the $i - 1$ closest nodes.
Finding the $i$th closest node repeatedly

An example

![Graph with nodes and edges labeled with distances]

- Node 0
- Node 1: 9
- Node 2: 6
- Node 3: 13
- Node 4: 30
- Node 5: 18
- Node 6: 10
- Node 7: 6
- Node 8: 20
- Node 9: 11
- Node 10: 19
- Node 11: 6
- Node 12: 25

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Spring 2019
Finding the $i$th closest node repeatedly

An example

$x = \{a, b, c, d, e\}$
Finding the $i$th closest node repeatedly

An example

$x_i = \{a, c\}$
Finding the $i$th closest node repeatedly

An example

$X = \{a, c, b\}$
Finding the $i$th closest node repeatedly

An example

$X = \{a, b, c, e\}$.
Finding the $i$th closest node repeatedly

An example
Finding the $i$th closest node repeatedly

An example

![Graph with labeled nodes and edges](image)
Finding the $i$th closest node repeatedly

An example
Finding the $i$th closest node repeatedly

An example
Finding the $i$th closest node

Corollary

*The $i$th closest node is adjacent to $X$.***
Finding the $i$th closest node

1. $X$ contains the $i - 1$ closest nodes to $s$
2. Want to find the $i$th closest node from $V - X$.

1. For each $u \in V - X$ let $P(s, u, X)$ be a shortest path from $s$ to $u$ using only nodes in $X$ as intermediate vertices.
2. Let $d'(s, u)$ be the length of $P(s, u, X)$
Finding the $i$th closest node

1. $X$ contains the $i - 1$ closest nodes to $s$

2. Want to find the $i$th closest node from $V - X$.

1. For each $u \in V - X$ let $P(s, u, X)$ be a shortest path from $s$ to $u$ using only nodes in $X$ as intermediate vertices.

2. Let $d'(s, u)$ be the length of $P(s, u, X)$

Observations: for each $u \in V - X$,

1. $\text{dist}(s, u) \leq d'(s, u)$ since we are constraining the paths

2. $d'(s, u) = \min_{t \in X}(\text{dist}(s, t) + \ell(t, u))$ - Why?
Finding the $i$th closest node

1. $X$ contains the $i - 1$ closest nodes to $s$
2. Want to find the $i$th closest node from $V - X$.

For each $u \in V - X$ let $P(s, u, X)$ be a shortest path from $s$ to $u$ using only nodes in $X$ as intermediate vertices.

2. Let $d'(s, u)$ be the length of $P(s, u, X)$

Observations: for each $u \in V - X$,

1. $\text{dist}(s, u) \leq d'(s, u)$ since we are constraining the paths
2. $d'(s, u) = \min_{t \in X} (\text{dist}(s, t) + \ell(t, u))$ - Why?

Lemma

If $v$ is the $i$th closest node to $s$, then $d'(s, v) = \text{dist}(s, v)$. 

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Finding the $i$th closest node

**Lemma**

Given:

1. $X$: Set of $i-1$ closest nodes to $s$.
2. $d'(s, u) = \min_{t \in X} (\text{dist}(s, t) + \ell(t, u))$

If $v$ is an $i$th closest node to $s$, then $d'(s, v) = \text{dist}(s, v)$.

**Proof.**

Let $v$ be the $i$th closest node to $s$. Then there is a shortest path $P$ from $s$ to $v$ that contains only nodes in $X$ as intermediate nodes (see previous claim). Therefore $d'(s, v) = \text{dist}(s, v)$. 

\[ \Box \]
Finding the $i$th closest node

**Lemma**

If $v$ is an $i$th closest node to $s$, then $d'(s, v) = \text{dist}(s, v)$.

**Corollary**

The $i$th closest node to $s$ is the node $v \in V - X$ such that $d'(s, v) = \min_{u \in V - X} d'(s, u)$.

**Proof.**

For every node $u \in V - X$, $\text{dist}(s, u) \leq d'(s, u)$ and for the $i$th closest node $v$, $\text{dist}(s, v) = d'(s, v)$. Moreover, $\text{dist}(s, u) \geq \text{dist}(s, v)$ for each $u \in V - X$.

$$d'(s, v) = \text{dist}(s, v) \leq \text{dist}(s, u) \leq d'(s, u)$$
Algorithm

Initialize for each node $v$: $\text{dist}(s, v) = \infty$

Initialize $X = \emptyset$, $d'(s, s) = 0$

for $i = 1$ to $|V|$ do

(* Invariant: $X$ contains the $i - 1$ closest nodes to $s$ *)

(* Invariant: $d'(s, u)$ is shortest path distance from $u$ to $s$ using only $X$ as intermediate nodes *)

Let $v$ be such that $d'(s, v) = \min_{u \in V - X} d'(s, u)$

$\text{dist}(s, v) = d'(s, v)$

$X = X \cup \{v\}$

for each node $u$ in $V - X$ do

$d'(s, u) = \min_{t \in X} \left( \text{dist}(s, t) + \ell(t, u) \right)$
Algorithm

Initialize for each node $v$: $\text{dist}(s, v) = \infty$

Initialize $X = \emptyset$, $d'(s, s) = 0$

for $i = 1$ to $|V|$ do

(* Invariant: $X$ contains the $i-1$ closest nodes to $s$ *)
(* Invariant: $d'(s, u)$ is shortest path distance from $u$ to $s$
  using only $X$ as intermediate nodes*)

Let $v$ be such that $d'(s, v) = \min_{u \in V - X} d'(s, u)$

$\text{dist}(s, v) = d'(s, v)$

$X = X \cup \{v\}$

for each node $u$ in $V - X$ do

$$d'(s, u) = \min_{t \in X} \left( \text{dist}(s, t) + \ell(t, u) \right)$$

Correctness: By induction on $i$ using previous lemmas.
Algorithm

Initialize for each node \( v \): \( \text{dist}(s, v) = \infty \)
Initialize \( X = \emptyset, \ d'(s, s) = 0 \)
for \( i = 1 \) to \( |V| \) do
  (* Invariant: \( X \) contains the \( i-1 \) closest nodes to \( s \) *)
  (* Invariant: \( d'(s, u) \) is shortest path distance from \( u \) to \( s \)
    using only \( X \) as intermediate nodes*)
  Let \( v \) be such that \( d'(s, v) = \min_{u \in V - X} d'(s, u) \)
  \( \text{dist}(s, v) = d'(s, v) \)
  \( X = X \cup \{ v \} \)
  for each node \( u \) in \( V - X \) do
    \( d'(s, u) = \min_{t \in X} \left( \text{dist}(s, t) + \ell(t, u) \right) \)

Correctness: By induction on \( i \) using previous lemmas.
Running time:
Algorithm

Initialize for each node $v$: $\text{dist}(s, v) = \infty$
Initialize $X = \emptyset$, $d'(s, s) = 0$

for $i = 1$ to $|V|$ do

(* Invariant: $X$ contains the $i - 1$ closest nodes to $s$ *)

(* Invariant: $d'(s, u)$ is shortest path distance from $u$ to $s$
using only $X$ as intermediate nodes*)

Let $v$ be such that $d'(s, v) = \min_{u \in V - X} d'(s, u)$

$\text{dist}(s, v) = d'(s, v)$

$X = X \cup \{v\}$

for each node $u$ in $V - X$ do

$d'(s, u) = \min_{t \in X} \left( \text{dist}(s, t) + \ell(t, u) \right)$

Correctness: By induction on $i$ using previous lemmas.
Running time: $O(n \cdot (n + m))$ time.

$n$ outer iterations. In each iteration, $d'(s, u)$ for each $u$ by
scanning all edges out of nodes in $X$; $O(m + n)$ time/iteration.
Example: Dijkstra algorithm in action

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![Dijkstra algorithm diagram with nodes and edges labeled with distances.]

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Improved Algorithm

1. Main work is to compute the $d'(s, u)$ values in each iteration.
2. $d'(s, u)$ changes from iteration $i$ to $i + 1$ only because of the node $v$ that is added to $X$ in iteration $i$. 

**Running time:** $O(m + n^2)$ time.
Main work is to compute the \( d'(s, u) \) values in each iteration.

\( d'(s, u) \) changes from iteration \( i \) to \( i + 1 \) only because of the node \( v \) that is added to \( X \) in iteration \( i \).

1. Initialize for each node \( v \), \( \text{dist}(s, v) = d'(s, v) = \infty \)
2. Initialize \( X = \emptyset \), \( d'(s, s) = 0 \)

\[
\text{for } i = 1 \text{ to } |V| \text{ do }
\]

\[
// X \text{ contains the } i - 1 \text{ closest nodes to } s, \\
// and the values of } d'(s, u) \text{ are current}
\]

- Let \( v \) be node realizing \( d'(s, v) = \min_{u \in V - X} d'(s, u) \)
- \( \text{dist}(s, v) = d'(s, v) \)
- \( X = X \cup \{v\} \)

- Update \( d'(s, u) \) for each \( u \) in \( V - X \) as follows:
  \[
  d'(s, u) = \min\left(d'(s, u), \text{dist}(s, v) + \ell(v, u)\right)
  \]

Running time:
\[
O\left(n^2 + \sum_{v \in V} \deg(v)\right) = O\left(n^2 + m\right)
\]
Improved Algorithm

Initialize for each node $v$, $\text{dist}(s, v) = d'(s, v) = \infty$

Initialize $X = \emptyset$, $d'(s, s) = 0$

for $i = 1$ to $|V|$ do

// $X$ contains the $i - 1$ closest nodes to $s$,
// and the values of $d'(s, u)$ are current
Let $v$ be node realizing $d'(s, v) = \min_{u \in V \setminus X} d'(s, u)$
$\text{dist}(s, v) = d'(s, v)$
$X = X \cup \{v\}$
Update $d'(s, u)$ for each $u$ in $V \setminus X$ as follows:
$$d'(s, u) = \min\left(d'(s, u), \text{dist}(s, v) + \ell(v, u)\right)$$

Running time: $O(m + n^2)$ time.

1. $n$ outer iterations and in each iteration following steps
2. updating $d'(s, u)$ after $v$ is added takes $O(\text{deg}(v))$ time so total work is $O(m)$ since a node enters $X$ only once
3. Finding $v$ from $d'(s, u)$ values is $O(n)$ time
Dijkstra’s Algorithm

1. eliminate $d'(s, u)$ and let $\text{dist}(s, u)$ maintain it
2. update $\text{dist}$ values after adding $v$ by scanning edges out of $v$

Initialize for each node $v$, $\text{dist}(s, v) = \infty$
Initialize $X = \emptyset$, $\text{dist}(s, s) = 0$
for $i = 1$ to $|V|$ do
    Let $v$ be such that $\text{dist}(s, v) = \min_{u \in V - X} \text{dist}(s, u)$
    $X = X \cup \{v\}$
    for each $u$ in $\text{Adj}(v)$ do
        $\text{dist}(s, u) = \min(\text{dist}(s, u), \text{dist}(s, v) + \ell(v, u))$

Priority Queues to maintain $\text{dist}$ values for faster running time
Dijkstra’s Algorithm

1. eliminate \(d'(s, u)\) and let \(\text{dist}(s, u)\) maintain it
2. update \(\text{dist}\) values after adding \(v\) by scanning edges out of \(v\)

Initialize for each node \(v\), \(\text{dist}(s, v) = \infty\)
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   Let \(v\) be such that \(\text{dist}(s, v) = \min_{u \in V - X} \text{dist}(s, u)\)
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   for each \(u\) in \(\text{Adj}(v)\) do
      \(\text{dist}(s, u) = \min(\text{dist}(s, u), \text{dist}(s, v) + \ell(v, u))\)

Priority Queues to maintain \(\text{dist}\) values for faster running time

1. Using heaps and standard priority queues: \(O((m + n) \log n)\)
2. Using Fibonacci heaps: \(O(m + n \log n)\).
Priority Queues

Data structure to store a set $S$ of $n$ elements where each element $v \in S$ has an associated real/integer key $k(v)$ such that the following operations:

1. **makePQ**: create an empty queue.
2. **findMin**: find the minimum key in $S$.
3. **extractMin**: Remove $v \in S$ with smallest key and return it.
4. **insert($v$, $k(v)$)**: Add new element $v$ with key $k(v)$ to $S$.
5. **delete($v$)**: Remove element $v$ from $S$.
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6. **decreaseKey**($v$, $k'(v)$): decrease key of $v$ from $k(v)$ (current key) to $k'(v)$ (new key). Assumption: $k'(v) \leq k(v)$.
7. **meld**: merge two separate priority queues into one.
Priority Queues

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7. **meld**: merge two separate priority queues into one.

All operations can be performed in $O(\log n)$ time. **decreaseKey** is implemented via **delete** and **insert**.
Dijkstra’s Algorithm using Priority Queues

\[ Q \leftarrow \text{makePQ}() \]
\[ \text{insert}(Q, (s, 0)) \]
\[ \text{for each node } u \neq s \text{ do} \]
\[ \text{insert}(Q, (u, \infty)) \]
\[ X \leftarrow \emptyset \]
\[ \text{for } i = 1 \text{ to } |V| \text{ do} \]
\[ (v, \text{dist}(s, v)) = \text{extractMin}(Q) \]
\[ X = X \cup \{v\} \]
\[ \text{for each } u \text{ in Adj}(v) \text{ do} \]
\[ \text{decreaseKey}(Q, (u, \min(\text{dist}(s, u), \text{dist}(s, v) + \ell(v, u)))) \]

Priority Queue operations:

1. \( O(n) \) \textbf{insert} operations
2. \( O(n) \) \textbf{extractMin} operations
3. \( O(m) \) \textbf{decreaseKey} operations

\[ n \log n + \sum_{u \in V} \deg(u) \log n \]
\[ (n + m) \log n \]
Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

1. All operations can be done in $O(\log n)$ time
Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

All operations can be done in $O(\log n)$ time

Dijkstra’s algorithm can be implemented in $O((n + m) \log n)$ time.
### Fibonacci Heaps

1. **extractMin, insert, delete, meld** in \( O(\log n) \) time
2. **decreaseKey** in \( O(1) \) amortized time:

---

**Relaxed Heaps:**

- **decreaseKey** in \( O(1) \) worst case time but at the expense of **meld** (not necessary for Dijkstra's algorithm)

---

**Dijkstra's algorithm** can be implemented in \( O(n \log n + m) \) time. If \( m = \Omega(n \log n) \), running time is linear in input size.

---

Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. **Rank-Pairing Heaps** (European Symposium on Algorithms, September 2009!)

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**Fibonacci Heaps**

1. **extractMin, insert, delete, meld** in $O(\log n)$ time
2. **decreaseKey** in $O(1)$ amortized time: $\ell$ decreaseKey operations for $\ell \geq n$ take together $O(\ell)$ time
3. Relaxed Heaps: **decreaseKey** in $O(1)$ worst case time but at the expense of **meld** (not necessary for Dijkstra’s algorithm)
### Fibonacci Heaps

1. **extractMin, insert, delete, meld** in $O(\log n)$ time

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1. Dijkstra’s algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.
Fibonacci Heaps

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Dijkstra’s algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.

Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)
Dijkstra’s algorithm finds the shortest path distances from $s$ to $V$. 

**Question:** How do we find the paths themselves?
Dijkstra’s algorithm finds the shortest path distances from $s$ to $V$.

**Question:** How do we find the paths themselves?

```plaintext
Q = makePQ()
insert(Q, (s, 0))
prev(s) ← null
for each node $u \neq s$ do
    insert(Q, (u, ∞))
    prev(u) ← null

X = ∅
for $i = 1$ to $|V|$ do
    ($v$, dist($s$, $v$)) = extractMin(Q)
    X = X ∪ {v}
    for each $u$ in Adj($v$) do
        if (dist($s$, $v$) + ℓ($v$, $u$) < dist($s$, $u$)) then
            decreaseKey(Q, (u, dist($s$, $v$) + ℓ($v$, $u$)))
            prev(u) = v
```
**Lemma**

The edge set \((u, \text{prev}(u))\) is the reverse of a shortest path tree rooted at \(s\). For each \(u\), the reverse of the path from \(u\) to \(s\) in the tree is a shortest path from \(s\) to \(u\).

**Proof Sketch.**

1. The edge set \(\{(u, \text{prev}(u)) \mid u \in V\}\) induces a directed in-tree rooted at \(s\) (Why?)

2. Use induction on \(|X|\) to argue that the tree is a shortest path tree for nodes in \(V\).
Dijkstra’s algorithm gives shortest paths from \( s \) to all nodes in \( V \). How do we find shortest paths from all of \( V \) to \( s \)?
Dijkstra’s algorithm gives shortest paths from \( s \) to all nodes in \( V \). How do we find shortest paths from all of \( V \) to \( s \)?

1. In undirected graphs shortest path from \( s \) to \( u \) is a shortest path from \( u \) to \( s \) so there is no need to distinguish.

2. In directed graphs, use Dijkstra’s algorithm in \( G^{\text{rev}} \)!
Shortest paths between sets of nodes

Suppose we are given $S \subset V$ and $T \subset V$. Want to find shortest path from $S$ to $T$ defined as:

$$\text{dist}(S, T) = \min_{s \in S, t \in T} \text{dist}(s, t)$$

How do we find $\text{dist}(S, T)$?
Example Problem

You want to go from your house to a friend’s house. Need to pick up some dessert along the way and hence need to stop at one of the many potential stores along the way. How do you calculate the “shortest” trip if you include this stop?
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Given $G = (V, E)$ and edge lengths $\ell(e), e \in E$. Want to go from $s$ to $t$. A subset $X \subset V$ that corresponds to stores. Want to find $\min_{x \in X} d(s, x) + d(x, t)$.
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**Basic solution:** Compute for each $x \in X$, $d(s, x)$ and $d(x, t)$ and take minimum. $2|X|$ shortest path computations. $O(|X|(m + n \log n))$. 
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Given \( G = (V, E) \) and edge lengths \( \ell(e), e \in E \). Want to go from \( s \) to \( t \). A subset \( X \subset V \) that corresponds to stores. Want to find \( \min_{x \in X} d(s, x) + d(x, t) \).

**Basic solution:** Compute for each \( x \in X \), \( d(s, x) \) and \( d(x, t) \) and take minimum. \( 2|X| \) shortest path computations. \( O(|X|(m + n \log n)) \).

**Better solution:** Compute shortest path distances from \( s \) to every node \( v \in V \) with one Dijkstra. Compute from every node \( v \in V \) shortest path distance to \( t \) with one Dijkstra.