Context Free Languages and Grammars

Lecture 7
Tuesday, February 5, 2019
Regular Languages

- Regular expressions allow us to describe/express a class of languages compactly and precisely.
- Equivalence with DFAs show the following: given any regular expression $r$ there is a very efficient algorithm for solving the language recognition problem for $L(r)$: given $w \in \Sigma^*$ is $w \in L(r)$?

In fact the running time of the algorithm is linear in $|w|$. 

Disadvantage of regular expressions/languages: too simple and cannot express interesting features such as balanced parenthesis that we need in programming languages. No recursion allowed even in limited form.
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Language classes: Chomsky Hierarchy

Generative models for languages based on grammars.

Regular  \rightarrow  Context Free  \rightarrow  Context Sensitive  \rightarrow  Recursively Enumerable  \rightarrow  All
For each class one can define a corresponding class of machines.
Regular vs. Context Free Languages

Regular Languages: Built from strings using:

1. Sequencing \( A \cdot B \)
2. Branching \( A + B \)
3. Repetition \( A^* \)
Regular vs. Context Free Languages

**Regular Languages:** Built from strings using:
1. Sequencing
2. Branching
3. Repetition

**Context Free Languages:** Built from strings using:
1. Sequencing
2. Branching
3. Recursion
What stack got to do with it?
What’s a stack but a second hand memory?

1. **DFA/NFA/Regular expressions.**
   - ≡ constant memory computation.

2. Turing machines **DFA/NFA + unbounded memory.**
   - ≡ a standard computer/program.
What stack got to do with it?

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1. **DFA/NFA/Regular expressions.**
   - constant memory computation.

2. **NFA + stack**
   - context free grammars (CFG).

3. Turing machines **DFA/NFA + unbounded memory.**
   - a standard computer/program.
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What’s a stack but a second hand memory?

1. **DFA/NFA/Regular expressions.**
   ≡ constant memory computation.

2. **NFA + stack**
   ≡ context free grammars (CFG).

3. Turing machines **DFA/NFA + unbounded memory.**
   ≡ a standard computer/program.
   ≡ **NFA** with two stacks.
**Question:** What is a valid C program? Or a Python program?

**Question:** Given a string $w$ what is an algorithm to check whether $w$ is a valid C program? The parsing problem.
Context Free Languages and Grammars

- Programming Language Specification
- Parsing
- Natural language understanding
- Generative model giving structure
- ...

CFLs provide a good balance between expressivity and tractability. Limited form of recursion.
<relational-expression> ::= <shift-expression>
    | <relational-expression> < <shift-expression>
    | <relational-expression> <= <shift-expression>
    | <relational-expression> >= <shift-expression>
    | <relational-expression> > <shift-expression>

<shift-expression> ::= <additive-expression>
    | <shift-expression> << <additive-expression>
    | <shift-expression> >> <additive-expression>

<additive-expression> ::= <multiplicative-expression>
    | <additive-expression> + <multiplicative-expression>
    | <additive-expression> - <multiplicative-expression>

<multiplicative-expression> ::= <cast-expression>
    | <multiplicative-expression> * <cast-expression>
    | <multiplicative-expression> / <cast-expression>
    | <multiplicative-expression> % <cast-expression>

<cast-expression> ::= <cast-expression>
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English sentences can be described as

\[ (S) \rightarrow (NP)(VP) \]
\[ (NP) \rightarrow (CN) | (CN)(PP) \]
\[ (VP) \rightarrow (CV) | (CV)(PP) \]
\[ (PP) \rightarrow (P)(CN) \]
\[ (CN) \rightarrow (A)(N) \]
\[ (CV) \rightarrow (V) | (V)(NP) \]
\[ (A) \rightarrow \text{a} | \text{the} \]
\[ (N) \rightarrow \text{boy} | \text{girl} | \text{flower} \]
\[ (V) \rightarrow \text{touched} | \text{likes} | \text{sees} \]
\[ (P) \rightarrow \text{with} \]

**English Sentences**

*Examples*

\[
\text{noun-phrs} \quad \text{verb-phrs}
\]
\[
a \quad \text{boy} \quad \text{sees}
\]
\[
\text{article} \quad \text{noun} \quad \text{verb}
\]

\[
\text{noun-phrs} \quad \text{verb-phrs}
\]
\[
\text{the} \quad \text{boy} \quad \text{sees} \quad \text{a} \quad \text{flower}
\]
\[
\text{article} \quad \text{noun} \quad \text{verb} \quad \text{noun-phrs}
\]
Models of Growth

- L-systems
- http://www.kevs3d.co.uk/dev/lsystems/
Kolam drawing generated by grammar
Definition

A Context Free Grammar (CFG) is a quadruple \( G = (V, T, P, S) \)

- \( V \) is a finite set of non-terminal symbols

\[ G = \left( \text{Variables, Terminals, Productions, Start var} \right) \]
A **CFG** is a quadruple $G = (V, T, P, S)$

- $V$ is a finite set of **non-terminal symbols**
- $T$ is a finite set of **terminal symbols** (alphabet)

Formally, $P \subset V \times (V \cup T)^*$. 

$S \in V$ is a start symbol.
A **CFG** is a quadruple $G = (V, T, P, S)$

- $V$ is a finite set of **non-terminal symbols**
- $T$ is a finite set of **terminal symbols** (alphabet)
- $P$ is a finite set of **productions**, each of the form $A \rightarrow \alpha$
  
  where $A \in V$ and $\alpha$ is a string in $(V \cup T)^*$. Formally, $P \subset V \times (V \cup T)^*$.

\[
G = \left( \begin{array}{c}
\text{Variables, } \\
\text{Terminals, } \\
\text{Productions, } \\
\text{Start var }
\end{array} \right)
\]
Definition

A **CFG** is a quadruple \( G = (V, T, P, S) \)

- \( V \) is a finite set of **non-terminal symbols**
- \( T \) is a finite set of **terminal symbols** (alphabet)
- \( P \) is a finite set of **productions**, each of the form \( A \rightarrow \alpha \)
  where \( A \in V \) and \( \alpha \) is a string in \((V \cup T)^*\).
  Formally, \( P \subset V \times (V \cup T)^* \).
- \( S \in V \) is a **start symbol**

\[ G = ( \text{Variables, Terminals, Productions, Start var} ) \]
Example

\[V = \{S\}\]

\[T = \{a, b\}\]

\[P = \{S \rightarrow \epsilon \mid a \mid b \mid aSa \mid bSb\}\]

(abbrev. for \(S \rightarrow \epsilon, S \rightarrow a, S \rightarrow b, S \rightarrow aSa, S \rightarrow bSb\))

What strings can \(S\) generate like this?
Example

- $V = \{S\}$
- $T = \{a, b\}$
- $P = \{S \rightarrow \epsilon \mid a \mid b \mid aSa \mid bSb\}$
  (abbrev. for $S \rightarrow \epsilon, S \rightarrow a, S \rightarrow b, S \rightarrow aSa, S \rightarrow bSb$)

\[
S \rightarrow aSa \rightarrow abSba \rightarrow abbSbba \rightarrow abbbba
\]

$abba bba$
Example

- $V = \{S\}$
- $T = \{a, b\}$
- $P = \{S \rightarrow \epsilon \mid a \mid b \mid aSa \mid bSb\}$
  (abbrev. for $S \rightarrow \epsilon, S \rightarrow a, S \rightarrow b, S \rightarrow aSa, S \rightarrow bSb$)

$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abbSbba \Rightarrow abb b bba$

What strings can $S$ generate like this?

Palindromes $\{a, b\}$

$\omega = \omega^R$
Example formally...

- $V = \{S\}$
- $T = \{a, b\}$
- $P = \{S \to \epsilon \mid a \mid b \mid aSa \mid bSb\}$
  (abbrev. for $S \to \epsilon, S \to a, S \to b, S \to aSa, S \to bSb$)

$$G = \left( \begin{array}{c} \{S\}, \{a, b\}, \{ \begin{array}{c} S \to \epsilon, \\ S \to a, \\ S \to b \\ S \to aSa \\ S \to bSb \end{array} \} \end{array} \right)$$
Palindromes

- Madam in Eden I’m Adam
- Dog doo? Good God!
- Dogma: I am God.
- A man, a plan, a canal, Panama
- Are we not drawn onward, we few, drawn onward to new era?
- http://www.palindromelist.net
Examples

\[ L = \{0^n1^n \mid n \geq 0\} \]

\[ G = (V, T, P, S) \]

\[ V = \{S\} \]

\[ T = \{0, 1\} \]

\[ P = \{S \rightarrow \epsilon, 0S1, 00S11, 0^n1^n\} \]
Examples

$L = \{0^n1^n \mid n \geq 0\}$

$S \rightarrow \epsilon \mid 0S1$
Notation and Convention

Let $G = (V, T, P, S)$ then

- $a, b, c, d, \ldots$, in $T$ (terminals)
- $A, B, C, D, \ldots$, in $V$ (non-terminals)
- $u, v, w, x, y, \ldots$ in $T^*$ for strings of terminals
- $\alpha, \beta, \gamma, \ldots$ in $(V \cup T)^*$
- $X, Y, X$ in $V \cup T$
"Derives" relation

Formalism for how strings are derived/generated

Definition

Let $G = (V, T, P, S)$ be a CFG. For strings $\alpha_1, \alpha_2 \in (V \cup T)^*$ we say $\alpha_1$ derives $\alpha_2$ denoted by $\alpha_1 \rightsquigarrow_G \alpha_2$ if there exist strings $\beta, \gamma, \delta$ in $(V \cup T)^*$ such that

1. $\alpha_1 = \beta A \delta$
2. $\alpha_2 = \beta \gamma \delta$
3. $A \rightarrow \gamma$ is in $P$.

Examples: $S \rightsquigarrow \epsilon$, $S \rightsquigarrow 0S1$, $0S1 \rightsquigarrow 00S11$, $0S1 \rightsquigarrow 01$. 

\[ \begin{align*}
\alpha_1 &= 0S1 \\
\alpha_2 &= 00S11 \\
S &\rightarrow 0S1
\end{align*} \]
For integer $k \geq 0$, $\alpha_1 \rightsquigarrow^k \alpha_2$ inductive defined:

- $\alpha_1 \rightsquigarrow^0 \alpha_2$ if $\alpha_1 = \alpha_2$
- $\alpha_1 \rightsquigarrow^k \alpha_2$ if $\alpha_1 \rightsquigarrow \beta_1$ and $\beta_1 \rightsquigarrow^{k-1} \alpha_2$. 

$\alpha_1 \rightsquigarrow^* \alpha_2$ if $\alpha_1 \rightsquigarrow^k \alpha_2$ for some $k$. 

Examples:

$S \rightsquigarrow^* \epsilon$, $0 S_1 \rightsquigarrow^* 0000011111$. 
Definition

For integer $k \geq 0$, $\alpha_1 \rightsquigarrow^k \alpha_2$ inductive defined:

- $\alpha_1 \rightsquigarrow^0 \alpha_2$ if $\alpha_1 = \alpha_2$
- $\alpha_1 \rightsquigarrow^k \alpha_2$ if $\alpha_1 \rightsquigarrow \beta_1$ and $\beta_1 \rightsquigarrow^{k-1} \alpha_2$.

Alternative definition: $\alpha_1 \rightsquigarrow^k \alpha_2$ if $\alpha_1 \rightsquigarrow^{k-1} \beta_1$ and $\beta_1 \rightsquigarrow \alpha_2$.

Examples: $S \rightsquigarrow^* \epsilon$, $S \rightsquigarrow^* 0008111$
“Derives” relation continued

**Definition**

For integer $k \geq 0$, $\alpha_1 \leadsto^k \alpha_2$ inductive defined:

- $\alpha_1 \leadsto^0 \alpha_2$ if $\alpha_1 = \alpha_2$
- $\alpha_1 \leadsto^k \alpha_2$ if $\alpha_1 \leadsto \beta_1$ and $\beta_1 \leadsto^{k-1} \alpha_2$.

Alternative definition: $\alpha_1 \leadsto^k \alpha_2$ if $\alpha_1 \leadsto^{k-1} \beta_1$ and $\beta_1 \leadsto \alpha_2$.

$\leadsto^*$ is the reflexive and transitive closure of $\leadsto$.

$\alpha_1 \leadsto^* \alpha_2$ if $\alpha_1 \leadsto^k \alpha_2$ for some $k$.

**Examples:** $S \leadsto^* \epsilon$, $0S1 \leadsto^* 0000011111$. 
Definition

The language generated by context-free grammar \( G = (V, T, P, S) \) is denoted by \( L(G) \) where \( L(G) = \{ w \in T^* \mid S \Rightarrow^* w \} \).
Context Free Languages

**Definition**

The language generated by CFG $G = (V, T, P, S)$ is denoted by $L(G)$ where $L(G) = \{ w \in T^* \mid S \xrightarrow{*} w \}$.

**Definition**

A language $L$ is context free (CFL) if it is generated by a context free grammar. That is, there is a CFG $G$ such that $L = L(G)$. 
Example

$L = \{0^n1^n \mid n \geq 0\}$

$L(G) \subseteq L, \quad L \subseteq L(G)$

* $\forall n, u = 0^n1^n$, there is $S \Rightarrow w$

$IH$: for $k < n, \quad S \Rightarrow 0^k1^k$

$s \Rightarrow 0^m1^n \Rightarrow 0^m1^n$

$s \Rightarrow 0^m1^n \Rightarrow 0^m1^n$

$S \Rightarrow 0^m1^n \Rightarrow 0^m1^n$

$s \Rightarrow 0^m1^n \Rightarrow 0^m1^n$

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$s \Rightarrow 0^m1^n \Rightarrow 0^m1^n$
Example

\[ L = \{0^n1^n | n \geq 0\} \quad S, \rightarrow \epsilon 10s,1 \]

\[ L = 0^*1^* \quad S \rightarrow \epsilon | 0s | 1s \]
Example

\[ L = \{0^n1^n \mid n \geq 0\} \]

\[ L = 0^*1^* \]

\[ L = \{0^n1^m \mid m > n\} = \{0^11^+ \} \]

\[
\begin{align*}
S & \rightarrow \varepsilon10S1 \\
S_2 & \rightarrow \varepsilon10S_21S_41 \\
S & \rightarrow 1S110S1 \\
S & \rightarrow A1 \\
A & \rightarrow \varepsilon10A1(A) \\
S & \rightarrow 811A1 \\
A & \rightarrow \varepsilon10A1
\end{align*}
\]
Example

$L = \{0^n1^n \mid n \geq 0\}$

$L = 0^*1^*$

$L = \{0^n1^m \mid m > n\}$  \hspace{1cm} S \rightarrow S|A, A \rightarrow 3|0A1

$L = \{0^n1^m \mid m < n\}$  \hspace{1cm} S \rightarrow OS|OA, A \rightarrow \epsilon|0A1
Example

$L = \{0^n1^n \mid n \geq 0\}$

$L = 0^*1^*$

$L = \{0^n1^m \mid m > n\}$

$L = \{0^n1^m \mid m < n\}$

$L = \{0^n1^m \mid m \neq n\}$

$P = \begin{cases} S \rightarrow B1C \\ B \rightarrow B1A1 \\ C \rightarrow OC10A \\ A \rightarrow \epsilon10A1 \end{cases}$

$V = \{S, A, B, C\}$

$T = \{0, 1\}$
Example

$L = \left\{ w \in \{(,\}\}^* \mid w \text{ is properly nested string of parenthesis} \right\}$

\[
S \rightarrow \varepsilon \mid (S) \mid (S)S \mid S(\)
\]

(can derive)

\[
( ( ) ( ) )
\]

\[
S \rightarrow (S) \rightarrow ( (S) ) \rightarrow ( ( ) ( ) )
\]

(cannot derive)

\[
(( ( ) ) ( ) )
\]
Example

\[ L = \{ w \in \{(,)\}^* \mid w \text{ is properly nested string of parenthesis} \} \]

\[ L = \{ w \in \{0, 1\}^* \mid w \text{ has equal number of 1s as 0's} \} \]

\[ S \rightarrow \varepsilon | S0S1 | S1S0 \]
\[ S \rightarrow \varepsilon | 0S1 | S0 | 1S \]
\[ S \rightarrow \varepsilon | 0S1 | S0 | 1S \]
\[ S \rightarrow 0S1, 1S0 \]
\[ S \rightarrow 1S0, 01S, S10, S01 \]
Closure Properties of CFLs

\[ G_1 = (V_1, T, P_1, S_1) \text{ and } G_2 = (V_2, T, P_2, S_2) \]

Assumption: \( V_1 \cap V_2 = \emptyset \), that is, non-terminals are not shared.
Closure Properties of CFLs

\[ G_1 = (V_1, T, P_1, S_1) \] and \[ G_2 = (V_2, T, P_2, S_2) \]

**Assumption:** \[ V_1 \cap V_2 = \emptyset \], that is, non-terminals are not shared

**Theorem**

CFLs are closed under union. \( L_1, L_2 \) CFLs implies \( L_1 \cup L_2 \) is a CFL.

**Theorem**

CFLs are closed under concatenation. \( L_1, L_2 \) CFLs implies \( L_1 \cdot L_2 \) is a CFL.

**Theorem**

CFLs are closed under Kleene star.

If \( L \) is a CFL \( \implies L^* \) is a CFL.
Closure Properties of CFLs

Union

$G_1 = (V_1, T, P_1, S_1)$ and $G_2 = (V_2, T, P_2, S_2)$

Assumption: $V_1 \cap V_2 = \emptyset$, that is, non-terminals are not shared.

Theorem

CFLs are closed under union. $L_1, L_2$ CFLs implies $L_1 \cup L_2$ is a CFL.
Theorem

CFLs are closed under concatenation. $L_1, L_2$ CFLs implies $L_1 \cdot L_2$ is a CFL.
Closure Properties of CFLs

Stardom (i.e., Kleene star)

Theorem

CFLs are closed under Kleene star.

If $L$ is a CFL $\Rightarrow L^*$ is a CFL.
Exercise

- Prove that every regular language is context-free using previous closure properties.
- Prove the set of regular expressions over an alphabet \( \Sigma \) forms a non-regular language which is context-free.
Theorem

CFLs are not closed under complement or intersection.

Theorem

If $L_1$ is a CFL and $L_2$ is regular then $L_1 \cap L_2$ is a CFL.
Canonical non-CFL

**Theorem**

\[ L = \{a^n b^n c^n \mid n \geq 0\} \text{ is not context-free.} \]

Proof based on **pumping lemma** for **CFLs**. Technical and outside the scope of this class.
Parse Trees or Derivation Trees

A tree to represent the derivation $S \Rightarrow^* w$.

- Rooted tree with root labeled $S$
- Non-terminals at each internal node of tree
- Terminals at leaves
- Children of internal node indicate how non-terminal was expanded using a production rule
Parse Trees or Derivation Trees

A tree to represent the derivation $S \xrightarrow{\star} w$.

- Rooted tree with root labeled $S$
- Non-terminals at each internal node of tree
- Terminals at leaves
- Children of internal node indicate how non-terminal was expanded using a production rule

A picture is worth a thousand words
A derivation tree for \textit{abbaab}
(also called “parse tree”)

A corresponding derivation of \textit{abbaab}

\[
S \rightarrow aSb \mid bSa \mid SS \mid ab \mid ba \mid \varepsilon
\]

\[
S \Rightarrow aSb \Rightarrow abS_\text{ab} \Rightarrow abS\text{S}_\text{ab} \Rightarrow ab\text{ba}_\text{Sab} \Rightarrow ab\text{baab}
\]
Ambiguity in CFLs

Definition

A CFG $G$ is ambiguous if there is a string $w \in L(G)$ with two different parse trees. If there is no such string then $G$ is unambiguous.

Example: $S \rightarrow S - S | 1 | 2 | 3$

\[
\begin{align*}
S &\rightarrow S - S & S &\rightarrow S - S \\
& \quad \quad \quad \quad S & & \quad \quad \quad \quad S \\
& \quad \quad \quad \quad 3 & \quad \quad \quad \quad 3 \\
& \quad \quad \quad \quad 2 & \quad \quad \quad \quad 2 \\
& \quad \quad \quad \quad 1 & \quad \quad \quad \quad 1 \\
& \quad \quad \quad \quad \quad \quad \quad \quad 3-(2-1) & \quad \quad \quad \quad \quad \quad \quad \quad (3-2)-1
\end{align*}
\]
Ambiguity in CFLs

- Original grammar: $S \rightarrow S \rightarrow S | 1 | 2 | 3$
- Unambiguous grammar:
  $S \rightarrow S - C | 1 | 2 | 3$
  $C \rightarrow 1 | 2 | 3$

The grammar forces a parse corresponding to left-to-right evaluation.

$3 - 2 - 1$
Inherently ambiguous languages

**Definition**

A **CFL** $L$ is inherently ambiguous if there is no unambiguous **CFG** $G$ such that $L = L(G)$.

There exist inherently ambiguous **CFLs**.

Example: $L = \{a^n b^m c^k | n = m \text{ or } m = k\}$.

Given a grammar $G$, it is undecidable to check whether $L(G)$ is inherently ambiguous. No algorithm!
Inherently ambiguous languages

Definition
A CFL $L$ is inherently ambiguous if there is no unambiguous CFG $G$ such that $L = L(G)$.

- There exist inherently ambiguous CFLs.
  
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Inductive proofs for CFGs

Question: How do we formally prove that a CFG $L(G) = L$?

Example: $S \rightarrow \epsilon | a | b | aSa | bSb$

Theorem: $L(G) = \{\text{palindromes}\} = \{w | w = w^R\}$
Inductive proofs for **CFGs**

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**Example**: $S \rightarrow \epsilon \mid a \mid b \mid aSa \mid bSb$

**Theorem**

$L(G) = \{\text{palindromes}\} = \{w \mid w = w^R\}$

Two directions:

- $L(G) \subseteq L$, that is, $S \xrightarrow{*} w$ then $w = w^R$
- $L \subseteq L(G)$, that is, $w = w^R$ then $S \xrightarrow{*} w$
Show that if $S \xrightarrow{*} w$ then $w = w^R$.

By induction on length of derivation, meaning:
For all $k \geq 1$, $S \xrightarrow{*^k} w$ implies $w = w^R$. 
Show that if $S \xrightarrow{\ast} w$ then $w = w^R$

By induction on length of derivation, meaning
For all $k \geq 1$, $S \xrightarrow{\ast}^k w$ implies $w = w^R$.

- If $S \xrightarrow{1} w$ then $w = \epsilon$ or $w = a$ or $w = b$. Each case $w = w^R$.
- Assume that for all $k < n$, that if $S \xrightarrow{k} w$ then $w = w^R$
- Let $S \xrightarrow{n} w$ (with $n > 1$). Wlog $w$ begin with $a$.
  - Then $S \rightarrow aSa \xrightarrow{k-1} aua$ where $w = aua$.
  - And $S \xrightarrow{n-1} u$ and hence IH, $u = u^R$.
  - Therefore $w^r = (aua)^R = (ua)^Ra = au^Ra = aua = w$. 

Chan, Har-Peled, Hassanieh (UIUC)
Show that if \( w = w^R \) then \( S \xrightarrow{*} w \).

By induction on \(|w|\)

That is, for all \( k \geq 0 \), \(|w| = k \) and \( w = w^R \) implies \( S \xrightarrow{*} w \).

**Exercise:** Fill in proof.
Mutual Induction

Situation is more complicated with grammars that have multiple non-terminals.

See Section 5.3.2 of the notes for an example proof.
Normal forms are a way to restrict form of production rules

**Advantage:** Simpler/more convenient algorithms and proofs
Normal Forms

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Two standard normal forms for CFGs
- Chomsky normal form
- Greibach normal form
Normal Forms

Chomsky Normal Form:

- Productions are all of the form $A \rightarrow BC$ or $A \rightarrow a$.
  - If $\epsilon \in L$ then $S \rightarrow \epsilon$ is also allowed.
- Every CFG $G$ can be converted into CNF form via an efficient algorithm
- Advantage: parse tree of constant degree.

Greibach Normal Form:

- Only productions of the form $A \rightarrow a\beta$ are allowed.
- All CFLs without $\epsilon$ have a grammar in GNF. Efficient algorithm.
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Algorithmic question: Given CFG $G$ and string $w \in \Sigma^*$ is $w \in L(G)$?

Later in course: algorithm for above problem that runs in $O(|w|^3)$ time for any fixed grammar $G$. Via dynamic programming.

Hence parsing problem for programming languages is solvable. However cubic time algorithm is too slow! For this reason grammars for PLs are restricted even further to make parsing algorithm faster (essentially linear time) — see CS 421 and compiler courses.

In programming languages some amount of “context” may be necessary. But CSL recognition is undecidable (no algorithm)! Hence people use ad hoc methods for the limited needs in PLs.
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Things to know: Pushdown Automata

**PDA:** a **NFA** coupled with a stack

**PDAs and CFGs** are equivalent: both generate exactly **CFLs**. **PDA** is a machine-centric view of **CFLs**.
Chomsky Hierarchy

See Wikipedia article for more on Chomsky Hierarchy including the grammar rules for Context Sensitive Languages etc.
https://en.wikipedia.org/wiki/Chomsky_hierarchy