Context Free Languages and Grammars

Lecture 7
Tuesday, February 5, 2019
Regular Languages

- Regular expressions allow us to describe/express a class of languages compactly and precisely.
- Equivalence with DFAs show the following: given any regular expression \( r \) there is a very efficient algorithm for solving the language recognition problem for \( L(r) \): given \( w \in \Sigma^* \) is \( w \in L(r) \)?

In fact the running time of the algorithm is linear in \(|w|\).

Disadvantage of regular expressions/languages: too simple and cannot express interesting features such as balanced parenthesis that we need in programming languages. No recursion allowed even in limited form.
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Disadvantage of regular expressions/languages: too simple and cannot express interesting features such as balanced parenthesis that we need in programming languages. No recursion allowed even in limited form.
Generative models for languages based on grammars.
For each class one can define a corresponding class of machines.
Regular vs. Context Free Languages

Regular Languages: Built from strings using:

1. Sequencing \( A \cdot B \)
2. Branching \( A + B \)
3. Repetition \( A^* \)
Regular vs. Context Free Languages

**Regular Languages**: Built from strings using:
1. Sequencing
2. Branching
3. Repetition

**Context Free Languages**: Built from strings using:
1. Sequencing
2. Branching
3. Recursion
What stack got to do with it?

What’s a stack but a second hand memory?

1. **DFA/NFA/Regular expressions.**
   \[ \equiv \text{constant memory computation.} \]

2. Turing machines **DFA/NFA** + unbounded memory.
   \[ \equiv \text{a standard computer/program.} \]
What stack got to do with it?
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1. **DFA/NFA/Regular expressions.**
   \[ \equiv \] constant memory computation.

2. **NFA + stack**
   \[ \equiv \] context free grammars (CFG).

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What stack got to do with it?
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1. **DFA/NFA/Regular expressions.**
   \[=\text{constant memory computation}.\]

2. **NFA + stack**
   \[=\text{context free grammars (CFG)}.\]

3. **Turing machines DFA/NFA + unbounded memory.**
   \[=\text{a standard computer/program}.\]
   \[=\text{NFA with two stacks}.\]
Question: What is a valid C program? Or a Python program?

Question: Given a string $w$ what is an algorithm to check whether $w$ is a valid C program? The parsing problem.
Context Free Languages and Grammars

- Programming Language Specification
- Parsing
- Natural language understanding
- Generative model giving structure
- ...

CFLs provide a good balance between expressivity and tractability. Limited form of recursion.
Programming Languages

<relational-expression> ::= <shift-expression>
  | <relational-expression> < <shift-expression>
  | <relational-expression> < <shift-expression>
  | <relational-expression> <= <shift-expression>
  | <relational-expression> >= <shift-expression>

<shift-expression> ::= <additive-expression>
  | <shift-expression> <<= <additive-expression>
  | <shift-expression> >>= <additive-expression>

<additive-expression> ::= <multiplicative-expression>
  | <additive-expression> + <multiplicative-expression>
  | <additive-expression> - <multiplicative-expression>

<multiplicative-expression> ::= <cast-expression>
  | <multiplicative-expression> * <cast-expression>
  | <multiplicative-expression> / <cast-expression>
  | <multiplicative-expression> % <cast-expression>

<cast-expression> ::= <unary-expression>
  | ( <type-name> ) <cast-expression>

<unary-expression> ::= <postfix-expression>
  | ++ <unary-expression>
  | -- <unary-expression>
  | <unary-operator> <cast-expression>
  | sizeof <unary-expression>
  | sizeof <type-name>

<postfix-expression> ::= <primary-expression>
  | <postfix-expression> [ <expression> ]
  | <postfix-expression> { <assignment-expression> }* 
  | <postfix-expression> . <identifier>
  | <postfix-expression> -> <identifier>
  | <postfix-expression> ++
  | <postfix-expression> --
English sentences can be described as

\[ (S) \rightarrow (NP)(VP) \]
\[ (NP) \rightarrow (CN) | (CN)(PP) \]
\[ (VP) \rightarrow (CV) | (CV)(PP) \]
\[ (PP) \rightarrow (P)(CN) \]
\[ (CN) \rightarrow (A)(N) \]
\[ (CV) \rightarrow (V) | (V)(NP) \]
\[ (A) \rightarrow \text{a} | \text{the} \]
\[ (N) \rightarrow \text{boy} | \text{girl} | \text{flower} \]
\[ (V) \rightarrow \text{touch} | \text{likes} | \text{sees} \]
\[ (P) \rightarrow \text{with} \]

---

**English Sentences**

*Examples*

- noun-phrs verb-phrs
  - a boy
    - sees
  - article noun verb

- noun-phrs verb-phrs
  - the boy sees a flower
  - article noun verb noun-phrs
Models of Growth

- $L$-systems
- http://www.kevs3d.co.uk/dev/lsystems/
Kolam drawing generated by grammar
A Context Free Grammar (CFG) is a quadruple $G = (V, T, P, S)$

- $V$ is a finite set of non-terminal symbols
- $T$ is a finite set of terminal symbols (alphabet)
- $P$ is a finite set of productions, each of the form $A \rightarrow \alpha$ where $A \in V$ and $\alpha$ is a string in $(V \cup T)^*$. Formally, $P \subset V \times (V \cup T)^*$.
- $S \in V$ is a start symbol
A **CFG** is a quadruple $G = (V, T, P, S)$

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Formally, $P \subseteq V \times (V \cup T)^*$. 

$S \in V$ is a **start symbol**

$$G = \left( \text{Variables, \; Terminals, \; Productions, \; Start var} \right)$$
A **CFG** is a quadruple $G = (V, T, P, S)$

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$G =$ (Variables, Terminals, Productions, Start var)
Context Free Grammar (CFG) Definition

A CFG is a quadruple \( G = (V, T, P, S) \)

- \( V \) is a finite set of **non-terminal symbols**
- \( T \) is a finite set of **terminal symbols** (alphabet)
- \( P \) is a finite set of **productions**, each of the form \( A \rightarrow \alpha \)
  where \( A \in V \) and \( \alpha \) is a string in \((V \cup T)^*\).
  Formally, \( P \subset V \times (V \cup T)^* \).
- \( S \in V \) is a **start symbol**

\[
G = \left( \text{Variables, Terminals, Productions, Start var} \right)
\]
Example

\[ V = \{ S \} \]

\[ T = \{ a, b \} \]

\[ P = \{ S \rightarrow \epsilon | a | b | aSa | bSb \} \]

(abbrev. for \( S \rightarrow \epsilon, S \rightarrow a, S \rightarrow b, S \rightarrow aSa, S \rightarrow bSb \))

What strings can \( S \) generate like this?
Example

- \( V = \{ S \} \)
- \( T = \{ a, b \} \)
- \( P = \{ S \rightarrow \epsilon \mid a \mid b \mid aSa \mid bSb \} \)
  
  (abbrev. for \( S \rightarrow \epsilon, S \rightarrow a, S \rightarrow b, S \rightarrow aSa, S \rightarrow bSb \))

\[
\begin{align*}
S \leadsto aSa & \leadsto abSba & \leadsto abbSbba & \leadsto abb b bba \\
\end{align*}
\]

\( abba bba \)
Example

- \( V = \{S\} \)
- \( T = \{a, b\} \)
- \( P = \{S \rightarrow \epsilon \mid a \mid b \mid aSa \mid bSb\} \)
  (abbrev. for \( S \rightarrow \epsilon, S \rightarrow a, S \rightarrow b, S \rightarrow aSa, S \rightarrow bSb\))

\[ S \rightsquigarrow aSa \rightsquigarrow abSba \rightsquigarrow abbSbba \rightsquigarrow abbbba \]

What strings can \( S \) generate like this?

Palindromes \( \{a, b\} \)

\( w = w^R \)
Example formally...

- $V = \{S\}$
- $T = \{a, b\}$
- $P = \{S \rightarrow \epsilon \mid a \mid b \mid aSa \mid bSb\}$
  (abbrev. for $S \rightarrow \epsilon, S \rightarrow a, S \rightarrow b, S \rightarrow aSa, S \rightarrow bSb$)

$$G = \left( \begin{array}{ccc} \{S\}, & \{a, b\}, & \left\{ \begin{array}{l} S \rightarrow \epsilon, \\ S \rightarrow a, \\ S \rightarrow b \\ S \rightarrow aSa \\ S \rightarrow bSb \end{array} \right\} \right)$$
Palindromes

- Madam in Eden I’m Adam
- Dog doo? Good God!
- Dogma: I am God.
- A man, a plan, a canal, Panama
- Are we not drawn onward, we few, drawn onward to new era?
- http://www.palindromelist.net
Examples

$L = \{0^n1^n \mid n \geq 0\}$

$G = (V, T, P, S)$

$V = \{S\}$

$T = \{0, 1\}$

$P = \{S \rightarrow \epsilon \mid 0 \sigma 1\}$

$S \rightarrow \epsilon$

$S \rightarrow 0S1 \rightarrow 01$

$S \rightarrow 0S1 \rightarrow 00S11 \rightarrow 0011$

$S \rightarrow 0^n1^n$
Examples

$L = \{0^n1^n \mid n \geq 0\}$

$S \rightarrow \epsilon \mid 0S1$
Let $G = (V, T, P, S)$ then

- $a, b, c, d, \ldots$, in $T$ (terminals)
- $A, B, C, D, \ldots$, in $V$ (non-terminals)
- $u, v, w, x, y, \ldots$ in $T^*$ for strings of terminals
- $\alpha, \beta, \gamma, \ldots$ in $(V \cup T)^*$
- $X, Y, X$ in $V \cup T$
“Derives” relation

Formalism for how strings are derived/generated

**Definition**

Let $G = (V, T, P, S)$ be a CFG. For strings $\alpha_1, \alpha_2 \in (V \cup T)^*$ we say $\alpha_1$ derives $\alpha_2$ denoted by $\alpha_1 \xrightarrow{G} \alpha_2$ if there exist strings $\beta, \gamma, \delta$ in $(V \cup T)^*$ such that

- $\alpha_1 = \beta \underline{A} \delta$
- $\alpha_2 = \beta \gamma \delta$
- $A \rightarrow \gamma$ is in $P$.

**Examples:** $S \xrightarrow{} \epsilon$, $S \xrightarrow{} 0S1$, $0S1 \xrightarrow{} 00S11$, $0S1 \xrightarrow{} 01$. 
“Derives” relation continued

**Definition**

For integer $k \geq 0$, $\alpha_1 \rightsquigarrow^k \alpha_2$ inductive defined:

- $\alpha_1 \rightsquigarrow^0 \alpha_2$ if $\alpha_1 = \alpha_2$

- $\alpha_1 \rightsquigarrow^k \alpha_2$ if $\alpha_1 \rightsquigarrow \beta_1$ and $\beta_1 \rightsquigarrow^{k-1} \alpha_2$.

Alternative definition:

$\alpha_1 \rightsquigarrow^k \alpha_2$ if $\alpha_1 \rightsquigarrow^{k-1} \beta_1$ and $\beta_1 \rightsquigarrow \alpha_2$.

$\alpha_1 \rightsquigarrow^* \alpha_2$ if $\alpha_1 \rightsquigarrow^k \alpha_2$ for some $k$.

Examples:

$S \rightsquigarrow^* \epsilon$, $0 S \rightsquigarrow^* 0000011111$. 
“Derives” relation continued

**Definition**

For integer $k \geq 0$, $\alpha_1 \rightsquigarrow^k \alpha_2$ inductive defined:

- $\alpha_1 \rightsquigarrow^0 \alpha_2$ if $\alpha_1 = \alpha_2$
- $\alpha_1 \rightsquigarrow^k \alpha_2$ if $\alpha_1 \rightsquigarrow \beta_1$ and $\beta_1 \rightsquigarrow^{k-1} \alpha_2$. 
- **Alternative definition:** $\alpha_1 \rightsquigarrow^k \alpha_2$ if $\alpha_1 \rightsquigarrow^{k-1} \beta_1$ and $\beta_1 \rightsquigarrow \alpha_2$

$S \rightsquigarrow^2 0008111$
"Derives" relation continued

**Definition**

For integer $k \geq 0$, $\alpha_1 \Rightarrow^k \alpha_2$ inductive defined:

- $\alpha_1 \Rightarrow^0 \alpha_2$ if $\alpha_1 = \alpha_2$
- $\alpha_1 \Rightarrow^k \alpha_2$ if $\alpha_1 \Rightarrow \beta_1$ and $\beta_1 \Rightarrow^{k-1} \alpha_2$.
- Alternative definition: $\alpha_1 \Rightarrow^k \alpha_2$ if $\alpha_1 \Rightarrow^{k-1} \beta_1$ and $\beta_1 \Rightarrow \alpha_2$

$\Rightarrow^*$ is the reflexive and transitive closure of $\Rightarrow$.

$\alpha_1 \Rightarrow^* \alpha_2$ if $\alpha_1 \Rightarrow^k \alpha_2$ for some $k$.

**Examples:** $S \Rightarrow^* \epsilon$, $0S1 \Rightarrow^* 0000011111$. 
Context Free Languages

Definition

The language generated by context-free grammar\( G = (V, T, P, S) \) is denoted by \( L(G) \) where \( L(G) = \{ w \in T^* | S \xrightarrow{*} w \} \).
Context Free Languages

Definition

The language generated by CFG $G = (V, T, P, S)$ is denoted by $L(G)$ where $L(G) = \{w \in T^* | S \Rightarrow^* w\}$.

Definition

A language $L$ is context free (CFL) if it is generated by a context free grammar. That is, there is a CFG $G$ such that $L = L(G)$.
Example

$L = \{0^n1^n \mid n \geq 0\}$

$L(\epsilon) = L$

$L(\epsilon) \subseteq L$, $L \subseteq L(\epsilon)$

\[ \forall n, w = 0^n1^n \text{, there is } S \xrightarrow{n} w \]

IH: For $k < n$, $S \xrightarrow{k} 0^k1^k$

$S \xrightarrow{0} 081 \xrightarrow{0} 00^m1^n$

$S \xrightarrow{\epsilon} 0^m1^n$

$S \xrightarrow{\epsilon} 0^m1^n$

$\forall w \in L(\epsilon); w = 01^n$ for some $n$

$S \xrightarrow{0} w \Rightarrow w = \epsilon = 01^n$

IH for any $k < n \Rightarrow S \xrightarrow{k} u \Rightarrow w = 0^k1^k$

let $w$ be an arbitrary string $S \in L(\epsilon) \Rightarrow S \xrightarrow{\epsilon} w$

$S \xrightarrow{\epsilon} u \Rightarrow S \xrightarrow{0} 081 \xrightarrow{0} 00^m1^n$

$S \xrightarrow{0} 0^m1^n$
Example

\[ L = \{0^n1^n \mid n \geq 0\} \]

\[ L = 0^*1^* \]

\[ S \rightarrow \varepsilon \mid 0S1 \]

\[ S \rightarrow \varepsilon \mid 0S1 \]

\[ S \rightarrow \varepsilon \mid 0S1 \]

\[ S \rightarrow \varepsilon \mid 0S1 \]
Example

\[ L = \{0^n1^n \mid n \geq 0\} \]

\[ L = 0^*1^* \]

\[ L = \{0^n1^m \mid m > n\} \]

\[ S_1 \rightarrow \varepsilon \mid 0S_1 \]

\[ S_2 \rightarrow \varepsilon \mid 0S_2 \mid S_4 \]

\[ S \rightarrow 1 \mid S1 \mid 0S1 \]

\[ S \rightarrow A1 \]

\[ A \rightarrow \varepsilon \mid 0A1 \mid (A) \]

\[ S \rightarrow 01 \mid A1 \]

\[ A \rightarrow \varepsilon \mid 0A1 \]
Example

\[ L = \{0^n1^n \mid n \geq 0\} \]

\[ L = 0^*1^* \]

\[ L = \{0^n1^m \mid m > n\} \quad S \rightarrow S1A1, \quad A \rightarrow 310A1 \]

\[ L = \{0^n1^m \mid m < n\} \quad S \rightarrow OS10A, \quad A \rightarrow e10A1 \]
Example

$L = \{0^n1^n | n \geq 0\}$

$L = 0^*1^*$

$L = \{0^n1^m | m > n\}$

$L = \{0^n1^m | m < n\}$

$L = \{0^n1^m | m \neq n\}$

$P = \begin{cases} S \rightarrow B1C \\ B \rightarrow B11A1 \\ C \rightarrow OC10A \\ A \rightarrow 210A1 \end{cases}$

$V = \{S,A,B,C\}$

$T = \{0,1\}$
Example

$L = \left\{ w \in \{(,)\}^* \mid w \text{ is properly nested string of parenthesis} \right\}$

\[
S \rightarrow \varepsilon \mid (s) \mid (s)S \mid S() \\
S \rightarrow \varepsilon \mid (s) \mid SS \\
S \rightarrow \varepsilon \mid (s) \mid (s)S
\]

can derive

\[
((())())
\]

cannot derive

\[
((())((())())
\]
Example

\[ L = \{ w \in \{ (, ) \}^* \mid w \text{ is properly nested string of parenthesis} \} \]

\[ L = \{ w \in \{ 0, 1 \}^* \mid w \text{ has equal number of 1s as 0's} \} \]

\[
S \rightarrow \varepsilon | S \sigma S | S | S \sigma S
\]

\[
S \rightarrow \varepsilon | \varepsilon | S \sigma S | S \sigma S | S \sigma S | S \sigma S
\]

\[
S \rightarrow \varepsilon | (S \sigma S) | (S \sigma S) | (S \sigma S) | (S \sigma S) | (S \sigma S) | (S \sigma S)
\]

\[
S \rightarrow 0 S 1, 1 S 0
\]

\[
S \rightarrow 1 S 0, 0 S 1, S 0 1, S 1 0
\]
Closure Properties of CFLs

\[ G_1 = (V_1, T, P_1, S_1) \text{ and } G_2 = (V_2, T, P_2, S_2) \]

Assumption: \( V_1 \cap V_2 = \emptyset \), that is, non-terminals are not shared
Closure Properties of CFLs

\[ G_1 = (V_1, T, P_1, S_1) \] and \[ G_2 = (V_2, T, P_2, S_2) \]

**Assumption:** \( V_1 \cap V_2 = \emptyset \), that is, non-terminals are not shared

**Theorem**

CFLs are closed under union. \( L_1, L_2 \) CFLs implies \( L_1 \cup L_2 \) is a CFL.

**Theorem**

CFLs are closed under concatenation. \( L_1, L_2 \) CFLs implies \( L_1 \cdot L_2 \) is a CFL.

**Theorem**

CFLs are closed under Kleene star. If \( L \) is a CFL \( \Rightarrow \) \( L^* \) is a CFL.
Closure Properties of CFLs

Union

\[ G_1 = (V_1, T, P_1, S_1) \] and \[ G_2 = (V_2, T, P_2, S_2) \]

Assumption: \( V_1 \cap V_2 = \emptyset \), that is, non-terminals are not shared.

Theorem

CFLs are closed under union. \( L_1, L_2 \) CFLs implies \( L_1 \cup L_2 \) is a CFL.

\[ G = \{V, T, P, S\} \]

\[ V = V_1 \cup V_2 \cup \{S\} \]

\[ T = \bar{T} \]

\[ P = P_1 \cup P_2 \cup \{S \rightarrow S, 1S_0\} \]

\[ S \rightarrow S \]
Closure Properties of CFLs

Concatenation

Theorem

CFLs are closed under concatenation. $L_1, L_2$ CFLs implies $L_1 \cdot L_2$ is a CFL.

$G = \{V, T, P, S\}$

$V = V_1 \cup V_2 \cup \{S\}$

$T = T$

$P = P \cup P_2 \cup \{S \rightarrow S_1 S_2\}$

$S = S$
Closure Properties of CFLs

Stardom (i.e., Kleene star)

**Theorem**

CFLs are closed under Kleene star.

If $L_i$ is a CFL $\implies L_i^*$ is a CFL.

$$G = \{V, T, \rho, S\}$$

$$V = V \cup \{S, T\}$$

$$T = T$$

$$\rho = \{S \rightarrow \varepsilon \mid SS, \} \cup \rho_i$$

$$S \rightarrow S$$
Exercise

- Prove that every regular language is context-free using previous closure properties.
- Prove the set of regular expressions over an alphabet $\Sigma$ forms a non-regular language which is context-free.

\[
\Sigma \cup \{+ , \cdot , \star , ( , ) , \varepsilon , \emptyset \}
\]

\[
\{ ( , ) \} \cup \{ + , \cdot , \star , ( , ) , \varepsilon , \emptyset \}
\]
Theorem

CFLs are not closed under complement or intersection.

Theorem

If $L_1$ is a CFL and $L_2$ is regular then $L_1 \cap L_2$ is a CFL.
Canonical non-CFL

Theorem

\[ L = \{a^n b^n c^n \mid n \geq 0\} \text{ is not context-free.} \]

Proof based on pumping lemma for CFLs. Technical and outside the scope of this class.
Parse Trees or Derivation Trees

A tree to represent the derivation $S \Rightarrow^* w$.

- Rooted tree with root labeled $S$
- Non-terminals at each internal node of tree
- Terminals at leaves
- Children of internal node indicate how non-terminal was expanded using a production rule
A tree to represent the derivation $S \Rightarrow^* w$.

- Rooted tree with root labeled $S$
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- Children of internal node indicate how non-terminal was expanded using a production rule

A picture is worth a thousand words
Example

A derivation tree for \textit{abbaab}
(also called “parse tree”)

\[ S \rightarrow aSb \mid bSa \mid SS \mid ab \mid ba \mid \varepsilon \]

A corresponding derivation of \textit{abbaab}

\[ S \rightarrow aSb \rightarrow abSSab \rightarrow abSSSab \rightarrow abbaSab \rightarrow abbaab \]
Ambiguity in CFLs

Definition

A CFG $G$ is ambiguous if there is a string $w \in L(G)$ with two different parse trees. If there is no such string then $G$ is unambiguous.

Example: $S \rightarrow S - S \mid 1 \mid 2 \mid 3$

```
3
S
--
S
3
S
--
S
2
1
```

$3 - (2 - 1) = 2$

```
3
S
--
S
3
S
--
S
2
```

$(3 - 2) - 1 = 0$
Ambiguity in CFLs

- Original grammar: \( S \rightarrow S - S \mid 1 \mid 2 \mid 3 \)
- Unambiguous grammar:
  \[
  S \rightarrow S - C \mid 1 \mid 2 \mid 3 \\
  C \rightarrow 1 \mid 2 \mid 3
  \]

The grammar forces a parse corresponding to left-to-right evaluation.
Inherently ambiguous languages

Definition

A CFL $L$ is inherently ambiguous if there is no unambiguous CFG $G$ such that $L = L(G)$. 

There exist inherently ambiguous CFLs. 

Example: $L = \{a^n b^m c^k | n = m \text{ or } m = k\}$ 

Given a grammar $G$, it is undecidable to check whether $L(G)$ is inherently ambiguous. No algorithm!
Inherently ambiguous languages

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A CFL $L$ is inherently ambiguous if there is no unambiguous CFG $G$ such that $L = L(G)$.

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- Example: $L = \{a^n b^m c^k \mid n = m \text{ or } m = k\}$
Inherently ambiguous languages

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A CFL $L$ is inherently ambiguous if there is no unambiguous CFG $G$ such that $L = L(G)$.

- There exist inherently ambiguous CFLs.
  Example: $L = \{a^n b^m c^k \mid n = m \text{ or } m = k\}$
- Given a grammar $G$ it is undecidable to check whether $L(G)$ is inherently ambiguous. No algorithm!
Inductive proofs for **CFGs**

**Question:** How do we formally prove that a **CFG** $L(G) = L$?

**Example:** $S \rightarrow \epsilon | a | b | aSa | bSb$

**Theorem**

$L(G) = \{\text{palindromes}\} = \{w | w = w^R\}$
Inductive proofs for **CFGs**

**Question:** How do we formally prove that a **CFG** $L(G) = L$?

**Example:** $S \rightarrow \epsilon \mid a \mid b \mid aSa \mid bSb$

**Theorem**

$L(G) = \{ \text{palindromes} \} = \{ w \mid w = w^R \}$

Two directions:

- $L(G) \subseteq L$, that is, $S \xrightarrow{*} w$ then $w = w^R$
- $L \subseteq L(G)$, that is, $w = w^R$ then $S \xrightarrow{*} w$
Show that if $S \Rightarrow^* w$ then $w = w^R$

By induction on length of derivation, meaning
For all $k \geq 1$, $S \Rightarrow^k w$ implies $w = w^R$. 
Show that if $S \Rightarrow^* w$ then $w = w^R$

By induction on length of derivation, meaning
For all $k \geq 1$, $S \Rightarrow^k w$ implies $w = w^R$.

- If $S \Rightarrow^1 w$ then $w = \epsilon$ or $w = a$ or $w = b$. Each case $w = w^R$.
- Assume that for all $k < n$, that if $S \Rightarrow^k w$ then $w = w^R$.
- Let $S \Rightarrow^n w$ (with $n > 1$). Wlog $w$ begin with $a$.
  - Then $S \Rightarrow aSa \Rightarrow^{k-1} aua$ where $w = aua$.
  - And $S \Rightarrow^{n-1} u$ and hence IH, $u = u^R$.
  - Therefore $w^r = (aua)^R = (ua)^Ra = au^Ra = aua = w$. 
\( L \subseteq L(G) \)

Show that if \( w = w^R \) then \( S \rightsquigarrow^* w \).

By induction on \( |w| \)
That is, for all \( k \geq 0 \), \( |w| = k \) and \( w = w^R \) implies \( S \rightsquigarrow^* w \).

Exercise: Fill in proof.
Mutual Induction

Situation is more complicated with grammars that have multiple non-terminals.

See Section 5.3.2 of the notes for an example proof.

\[
\begin{align*}
S & \rightarrow \varepsilon | 0A \\
A & \rightarrow \varepsilon | 1 | S \\
\end{align*}
\]

mutal \quad L(S) = \\
L(A) =
Normal forms are a way to restrict form of production rules

**Advantage:** Simpler/more convenient algorithms and proofs
Normal forms are a way to restrict form of production rules

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Two standard normal forms for **CFGs**
- Chomsky normal form
- Greibach normal form
Chomsky Normal Form:

- Productions are all of the form $A \rightarrow BC$ or $A \rightarrow a$.
  - If $\epsilon \in L$ then $S \rightarrow \epsilon$ is also allowed.
- Every CFG $G$ can be converted into CNF form via an efficient algorithm.
- Advantage: parse tree of constant degree.
Normal Forms

Chomsky Normal Form:
- Productions are all of the form $A \rightarrow BC$ or $A \rightarrow a$.
  If $\epsilon \in L$ then $S \rightarrow \epsilon$ is also allowed.
- Every CFG $G$ can be converted into CNF form via an efficient algorithm.
- Advantage: parse tree of constant degree.

Greibach Normal Form:
- Only productions of the form $A \rightarrow a\beta$ are allowed.
- All CFLs without $\epsilon$ have a grammar in GNF. Efficient algorithm.
- Advantage: Every derivation adds exactly one terminal.
Algorithmic question: Given CFG $G$ and string $w \in \Sigma^*$ is $w \in L(G)$?
Language recognition for CFLs

**Algorithmic question:** Given CFG $G$ and string $w \in \Sigma^*$ is $w \in L(G)$?

Later in course: algorithm for above problem that runs in $O(|w|^3)$ time for any fixed grammar $G$. Via dynamic programming.

Hence parsing problem for programming languages is solvable.
However cubic time algorithm is too slow! For this reason grammars for PLs are restricted even further to make parsing algorithm faster (essentially linear time) — see CS 421 and compiler courses.

In programming languages some amount of “context” may be necessary. But CSL recognition is undecidable (no algorithm)! Hence people use ad hoc methods for the limited needs in PLs.
PDA: a NFA coupled with a stack

PDA$s$ and CFG$s$ are equivalent: both generate exactly CFL$s$. PDA is a machine-centric view of CFL$s$. 
Chomsky Hierarchy

See Wikipedia article for more on Chomsky Hierarchy including the grammar rules for Context Sensitive Languages etc.
https://en.wikipedia.org/wiki/Chomsky_hierarchy