Non-deterministic Finite Automata (NFAs)

Lecture 4
Thursday, January 24, 2019
Part I

NFA Introduction
Non-deterministic Finite State Automata (NFAs)

Differences from DFA:
- From state $q$ on same letter $a \in \Sigma$, multiple possible states
- No transitions from $q$ on some letters
- $\varepsilon$-transitions!

Questions:
- Is this a "real" machine?
- What does it do?
Non-deterministic Finite State Automata (NFAs)

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- Is this a “real” machine?
- What does it do?
Machine on input string $w$ from state $q$ can lead to set of states (could be empty)
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- From $q_\varepsilon$ on 1
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- From $q_\varepsilon$ on 1
- From $q_\varepsilon$ on 0 $\{q_\varepsilon, q_0, q_{00}\}$
Machine on input string $w$ from state $q$ can lead to set of states (could be empty)

- From $q_\varepsilon$ on 1
- From $q_\varepsilon$ on 0
- From $q_0$ on $\varepsilon$ \{ $q_0$, $q_{00}$ \}
Machine on input string $w$ from state $q$ can lead to set of states (could be empty)

- From $q_{\epsilon}$ on 1
- From $q_{\epsilon}$ on 0
- From $q_0$ on $\epsilon$
- From $q_{\epsilon}$ on 01
NFA behavior

Machine on input string $w$ from state $q$ can lead to set of states (could be empty)

- From $q_\varepsilon$ on 1
- From $q_\varepsilon$ on 0
- From $q_0$ on $\varepsilon$
- From $q_\varepsilon$ on 01
- From $q_{00}$ on 00
Informal definition: An NFA $N$ accepts a string $w$ iff some accepting state is reached by $N$ from the start state on input $w$. 
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The language accepted (or recognized) by a NFA $N$ is denote by $L(N)$ and defined as: $L(N) = \{ w \mid N \text{ accepts } w \}$.
NFA acceptance: example

Is 01 accepted?

Comment: Unlike DFAs, it is easier in NFAs to show that a string is accepted than to show that a string is not accepted.
Is 01 accepted? ✓
Is 001 accepted? ✓
Is 01 accepted?
Is 001 accepted?
Is 100 accepted?
NFA acceptance: example

- Is \textbf{01} accepted?
- Is \textbf{001} accepted?
- Is \textbf{100} accepted?
- Are all strings in \textbf{1*01} accepted?
NFA acceptance: example

\[ \Sigma = \{0, 1\} \]

- Is 01 accepted?
- Is 001 accepted?
- Is 100 accepted?
- Are all strings in \(1^*01\) accepted?
- What is the language accepted by \(N\)?

\[(0+1)^*0(0+\epsilon)1(0+1)^*\]
NFA acceptance: example

- Is 01 accepted?
- Is 001 accepted?
- Is 100 accepted?
- Are all strings in 1*01 accepted?
- What is the language accepted by $N$?
Is \textbf{01} accepted?

Is \textbf{001} accepted?

Is \textbf{100} accepted?

Are all strings in \textbf{1*01} accepted?

What is the language accepted by \textbf{N}?

\textbf{Comment:} Unlike \textbf{DFA}s, it is easier in \textbf{NFA}s to show that a string is accepted than to show that a string is \textbf{not} accepted.
Simulating NFA

Example the first

(N1) A → B → C → D → E

Run it on input $ababa$.

Idea: Keep track of the states where the NFA might be at any given time.
Simulating NFA

Example the first

$t = 0$:

Remaining input: \textit{ababa}.
Simulating NFA

Example the first

$t = 0$: 

Remaining input: \textit{ababa}.

$t = 1$: 

Remaining input: \textit{baba}.
Simulating NFA
Example the first

$t = 1$:

Remaining input: *baba*.
Simulating NFA

Example the first

$t = 1$:

Remaining input: \textit{baba}.

$t = 2$:

Remaining input: \textit{aba}.
Simulating NFA

Example the first

$t = 2$:

Remaining input: $aba$. 

A \rightarrow B \rightarrow C \rightarrow D \rightarrow E
Simulating NFA

Example the first

$t = 2$:

Remaining input: $aba$.

$t = 3$:

Remaining input: $ba$. 

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Simulating NFA

Example the first

\[ t = 3: \]

\[ A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{a} D \xrightarrow{b} E \]

Remaining input: \textit{ba}.  

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Simulating NFA

Example the first

$t = 3$:

Remaining input: $ba$.

$t = 4$:

Remaining input: $a$. 
Simulating NFA

Example the first

$t = 4$:

A \rightarrow B \rightarrow C \rightarrow D \rightarrow E

Remaining input: $a$. 
Simulating NFA

Example the first

$t = 4$:

Remaining input: $a$.

$t = 5$:

Remaining input: $\varepsilon$. 
Simulating NFA

Example the first

$t = 5$:

\[ \begin{array}{cccccc}
A & \quad \text{a,b} \quad & B & \quad \text{a} \quad & C & \quad \text{b} \quad & D & \quad \text{a} \quad & E & \quad \text{b} \\
\end{array} \]

Remaining input: $\varepsilon$.

Accepts: \textit{ababa}. 

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A non-deterministic finite automata (NFA) $N = (Q, \Sigma, \delta, s, A)$ is a five tuple where

- $Q$ is a finite set whose elements are called states,
- $\Sigma$ is a finite set called the input alphabet,
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \rightarrow P(Q)$ is the transition function (here $P(Q)$ is the power set of $Q$),
- $s \in Q$ is the start state,
- $A \subseteq Q$ is the set of accepting/final states.

$\delta(q, a)$ for $a \in \Sigma \cup \{\epsilon\}$ is a subset of $Q$ — a set of states.
Reminder: Power set

For a set \( Q \) its power set is: \( P(Q) = 2^Q = \{ X \mid X \subseteq Q \} \) is the set of all subsets of \( Q \).

Example

\( Q = \{1, 2, 3, 4\} \)

\[
P(Q) = \left\{ \{1, 2, 3, 4\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1\}, \{2\}, \{3\}, \{4\}, \{} \right\}
\]
Example

\[ Q = \{ q_\varepsilon, q_0, q_{00}, q_p \} \]

\[ \Sigma = \{ 0, 1 \} \]

\[ \delta \]

\[ A = \{ q_p \} \]
Example

\[ Q = \{ q_\epsilon, q_0, q_{00}, q_p \} \]
Example

$Q = \{ q_\varepsilon, q_0, q_{00}, q_p \}$

$\Sigma =$
Example

\[ Q = \{q_ε, q_0, q_{00}, q_p\} \]

\[ Σ = \{0, 1\} \]
Example

- $Q = \{ q_\varepsilon, q_0, q_{00}, q_p \}$
- $\Sigma = \{ 0, 1 \}$
- $\delta$
Example

- $Q = \{ q_\varepsilon, q_0, q_{00}, q_p \}$
- $\Sigma = \{ 0, 1 \}$
- $\delta$
- $s =$
Example

- \( Q = \{ q_\varepsilon, q_0, q_{00}, q_p \} \)
- \( \Sigma = \{ 0, 1 \} \)
- \( \delta \)
- \( s = q_\varepsilon \)
Example

\[ Q = \{ q_\varepsilon, q_0, q_{00}, q_p \} \]

\[ \Sigma = \{ 0, 1 \} \]

\[ \delta \]

\[ s = q_\varepsilon \]

\[ A = \]
Example

- \( Q = \{ q_\epsilon, q_0, q_{00}, q_p \} \)
- \( \Sigma = \{ 0, 1 \} \)
- \( \delta \)
- \( s = q_\epsilon \)
- \( A = \{ q_p \} \)
Example
Transition function in detail...

\[
\begin{align*}
\delta(q_\varepsilon, \varepsilon) &= \\
\delta(q_\varepsilon, 0) &= \\
\delta(q_\varepsilon, 1) &= \\
\delta(q_0, \varepsilon) &= \\
\delta(q_0, 0) &= \\
\delta(q_0, 1) &= \\
\delta(q_{00}, \varepsilon) &= \\
\delta(q_{00}, 0) &= \\
\delta(q_{00}, 1) &= \\
\delta(q_p, \varepsilon) &= \\
\delta(q_p, 0) &= \\
\delta(q_p, 1) &= 
\end{align*}
\]
Example
Transition function in detail...

\[
\begin{align*}
\delta(q_\varepsilon, \varepsilon) &= \{q_\varepsilon\} \\
\delta(q_\varepsilon, 0) &= \{q_\varepsilon, q_0\} \\
\delta(q_\varepsilon, 1) &= \{q_\varepsilon\} \\
\delta(q_0, \varepsilon) &= \{q_0, q_{00}\} \\
\delta(q_0, 0) &= \{q_{00}\} \\
\delta(q_0, 1) &= \{\} \\
\delta(q_{00}, \varepsilon) &= \{q_{00}\} \\
\delta(q_{00}, 0) &= \{\} \\
\delta(q_{00}, 1) &= \{q_p\} \\
\delta(q_p, \varepsilon) &= \{q_p\} \\
\delta(q_p, 0) &= \{q_p\} \\
\delta(q_p, 1) &= \{q_p\}
\end{align*}
\]
Extending the transition function to strings

1. NFA \( N = (Q, \Sigma, \delta, s, A) \)
Extending the transition function to strings

1. **NFA** \( N = (Q, \Sigma, \delta, s, A) \)

2. \( \delta(q, a) \): set of states that \( N \) can go to from \( q \) on reading \( a \in \Sigma \cup \{\varepsilon\} \).
Extending the transition function to strings

1. NFA $N = (Q, \Sigma, \delta, s, A)$

2. $\delta(q, a)$: set of states that $N$ can go to from $q$ on reading $a \in \Sigma \cup \{\varepsilon\}$.

3. Want transition function $\delta^* : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$
Extending the transition function to strings

1. **NFA** $N = (Q, \Sigma, \delta, s, A)$

2. $\delta(q, a)$: set of states that $N$ can go to from $q$ on reading $a \in \Sigma \cup \{\varepsilon\}$.

3. Want transition function $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$

4. $\delta^*(q, w)$: set of states reachable on input $w$ starting in state $q$. 
Extending the transition function to strings

**Definition**

For NFA $N = (Q, \Sigma, \delta, s, A)$ and $q \in Q$ the $\epsilon$-reach($q$) is the set of all states that $q$ can reach using only $\epsilon$-transitions.

For $\epsilon$-reach(s) $= \{s, d, a\}$
Extending the transition function to strings

**Definition**

For NFA $N = (Q, \Sigma, \delta, s, A)$ and $q \in Q$ the $\epsilon$-reach($q$) is the set of all states that $q$ can reach using only $\epsilon$-transitions.

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Extending the transition function to strings

**Definition**

For NFA \( N = (Q, \Sigma, \delta, s, A) \) and \( q \in Q \) the \( \epsilon \text{reach}(q) \) is the set of all states that \( q \) can reach using only \( \epsilon \)-transitions.

**Definition**

Inductive definition of \( \delta^*: Q \times \Sigma^* \rightarrow \mathcal{P}(Q) \):

- if \( w = \epsilon \), \( \delta^*(q, w) = \epsilon \text{reach}(q) \)
Extending the transition function to strings

**Definition**
For \( NFA \ N = (Q, \Sigma, \delta, s, A) \) and \( q \in Q \) the \( \epsilon \text{reach}(q) \) is the set of all states that \( q \) can reach using only \( \epsilon \)-transitions.

**Definition**
Inductive definition of \( \delta^* : Q \times \Sigma^* \rightarrow \mathcal{P}(Q) \):
- if \( w = \epsilon \), \( \delta^*(q, w) = \epsilon \text{reach}(q) \)
- if \( w = a \) where \( a \in \Sigma \)
  \[
  \delta^*(q, a) = \bigcup_{p \in \epsilon \text{reach}(q)} \left( \bigcup_{r \in \delta(p, a)} \epsilon \text{reach}(r) \right)
  \]
Extending the transition function to strings

**Definition**

For NFA $N = (Q, \Sigma, \delta, s, A)$ and $q \in Q$ the $\epsilon\text{reach}(q)$ is the set of all states that $q$ can reach using only $\epsilon$-transitions.

**Definition**

Inductive definition of $\delta^* : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$:

- if $w = \epsilon$, $\delta^*(q, w) = \epsilon\text{reach}(q)$
- if $w = a$ where $a \in \Sigma$
  $$\delta^*(q, a) = \bigcup_{p \in \epsilon\text{reach}(q)} \bigcup_{r \in \delta(p, a)\epsilon\text{reach}(r)}$$
- if $w = ax$, $\delta^*(q, w) = \bigcup_{p \in \epsilon\text{reach}(q)} \bigcup_{r \in \delta(p, a)\delta^*(r, x)}$
Formal definition of language accepted by $N$

**Definition**

A string $w$ is accepted by NFA $N$ if $\delta_N^*(s, w) \cap A \neq \emptyset$.

**Definition**

The language $L(N)$ accepted by a NFA $N = (Q, \Sigma, \delta, s, A)$ is

$$\{ w \in \Sigma^* | \delta^*(s, w) \cap A \neq \emptyset \}.$$
A string $w$ is accepted by NFA $N$ if $\delta^*_N(s, w) \cap A \neq \emptyset$.

The language $L(N)$ accepted by a NFA $N = (Q, \Sigma, \delta, s, A)$ is

$$\{ w \in \Sigma^* \mid \delta^*(s, w) \cap A \neq \emptyset \}.$$
What is:

- $\delta^*(s, \varepsilon) = \varepsilon\text{-reach}(s) = \{s, d, a\}$
Example

What is:

- $\delta^*(s, \epsilon) = \{s, d, a\}$
- $\delta^*(s, 0) = \{s, b, d; a\}$
Example

What is:

- $\delta^*(s, \varepsilon)$
- $\delta^*(s, 0)$
- $\delta^*(c, 0)$

$\delta^*(c, 0) = \{c, g, d, a\}$

Proofs/oracles.

Metaphor is that if an NFA reads a string that is not in its language, it destroys something, but as universes.
Example

What is:

- $\delta^*(s, \epsilon)$
- $\delta^*(s, 0)$
- $\delta^*(c, 0)$
- $\delta^*(b, 00) = \{a, b\}$

$\epsilon$-reach (b) = \{b\}

$\epsilon$-reach (c) = \{c\}

$\epsilon$-reach (g) = \{g, d, f, c\}
Another definition of computation

**Definition**

$q \xrightarrow{w}_N p$: State $p$ of NFA $N$ is **reachable** from $q$ on $w$ \iff there exists a sequence of states $r_0, r_1, \ldots, r_k$ and a sequence $x_1, x_2, \ldots, x_k$ where $x_i \in \Sigma \cup \{\epsilon\}$, for each $i$, such that:

- $r_0 = q$,
- for each $i$, $r_{i+1} \in \delta(r_i, x_{i+1})$,
- $r_k = p$, and
- $w = x_1x_2x_3\cdots x_k$.

**Definition**

$\delta^* N(q, w) = \left\{ p \in Q \mid q \xrightarrow{w}_N p \right\}$. 
Why non-determinism?

- Non-determinism adds power to the model; richer programming language and hence (much) easier to “design” programs.
- Fundamental in theory to prove many theorems.
- Very important in practice directly and indirectly.
- Many deep connections to various fields in Computer Science and Mathematics.

Many interpretations of non-determinism. Hard to understand at the outset. Get used to it and then you will appreciate it slowly.
Part II

Constructing NFAs
Every **DFA** is a **NFA** so **NFA**s are at least as powerful as **DFAs**.

**NFA**s prove ability to “guess and verify” which simplifies design and reduces number of states

Easy proofs of some closure properties
Example

Strings that represent decimal numbers.
Example

Strings that represent decimal numbers.
Example

\{ strings that contain CS374 as a substring \}

\[ \Sigma = \{ 0, 1, 2, \ldots, 9, a, \ldots, z, A, \ldots, Z \} \]

\[ \Sigma = \{ \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z} \} \]
Example

- \{\text{strings that contain CS374 as a substring}\}
- \{\text{strings that contain CS374 or CS473 as a substring}\}
Example

- \{\text{strings that contain CS374 as a substring}\}
- \{\text{strings that contain CS374 or CS473 as a substring}\}
- \{\text{strings that contain CS374 and CS473 as substrings}\}
Example

$L_k = \{\text{bitstrings that have a } 1 \text{ } k \text{ positions from the end}\}$

DFA: Remember last $k$ bits $\Rightarrow 2^k$ states

NFA: $k$ states.

\[ \begin{array}{c}
\cdots \\
(0,1) \\
(0,1) \\
(0,1) \\
(0,1) \\
\end{array} \]
A simple transformation

**Theorem**

For every NFA $N$ there is another NFA $N'$ such that $L(N) = L(N')$ and such that $N'$ has the following two properties:

- $N'$ has single final state $f$ that has no outgoing transitions
- The start state $s$ of $N$ is different from $f$
Part III

Closure Properties of NFAs
Closure properties of NFAs

Are the class of languages accepted by NFAs closed under the following operations?

- union
- intersection
- concatenation
- Kleene star
- complement
Closure under union

**Theorem**

For any two NFA's $N_1$ and $N_2$ there is a NFA $N$ such that $L(N) = L(N_1) \cup L(N_2)$.
Closure under union

**Theorem**

For any two NFAs $N_1$ and $N_2$ there is a NFA $N$ such that $L(N) = L(N_1) \cup L(N_2)$.
Closure under concatenation

**Theorem**

For any two NFA s $N_1$ and $N_2$ there is a NFA $N$ such that $L(N) = L(N_1) \cdot L(N_2)$. 
Theorem

For any two NFA s $N_1$ and $N_2$ there is a NFA $N$ such that $L(N) = L(N_1) \cdot L(N_2)$.
Closure under Kleene star

**Theorem**

*For any NFA $N_1$ there is a NFA $N$ such that $L(N) = (L(N_1))^*$.*
Closure under Kleene star

**Theorem**

For any NFA $N_1$ there is a NFA $N$ such that $L(N) = (L(N_1))^*$. 
Theorem

For any NFA $N_1$ there is a NFA $N$ such that $L(N) = (L(N_1))^\ast$. 

$\epsilon$

$N_1$

$q_1$

Does not work! Why?
Closure under Kleene star

**Theorem**

For any NFA $N_1$ there is a NFA $N$ such that $L(N) = (L(N_1))^*$.

Does not work! Why?
Closure under Kleene star

Theorem

For any NFA $N_1$ there is a NFA $N$ such that $L(N) = (L(N_1))^*$. 
Part IV

NFA\textsuperscript{s} capture Regular Languages
### Regular Languages Recap

**Regular Languages**

- $\emptyset$ regular
- $\{\epsilon\}$ regular
- $\{a\}$ regular for $a \in \Sigma$
- $R_1 \cup R_2$ regular if both are
- $R_1R_2$ regular if both are
- $R^*$ is regular if $R$ is

**Regular Expressions**

- $\emptyset$ denotes $\emptyset$
- $\epsilon$ denotes $\{\epsilon\}$
- $a$ denote $\{a\}$
- $r_1 + r_2$ denotes $R_1 \cup R_2$
- $r_1r_2$ denotes $R_1R_2$
- $r^*$ denote $R^*$

Regular expressions denote regular languages — they explicitly show the operations that were used to form the language.
Theorem

For every regular language $L$ there is an NFA $N$ such that $L = L(N)$. 
NFA and Regular Language

Theorem

For every regular language $L$ there is an NFA $N$ such that $L = L(N)$.

Proof strategy:

- For every regular expression $r$ show that there is a NFA $N$ such that $L(r) = L(N)$
- Induction on length of $r$
NFA and Regular Language

For every regular expression $r$ show that there is a NFA $N$ such that $L(r) = L(N)$

Induction on length of $r$

Base cases: $\emptyset$, $\{\varepsilon\}$, $\{a\}$ for $a \in \Sigma$. 
NFA s and Regular Language

- For every regular expression $r$ show that there is a NFA $N$ such that $L(r) = L(N)$
- Induction on length of $r$

**Inductive cases:**
- $r_1, r_2$ regular expressions and $r = r_1 + r_2$. 

By induction there are NFAs $N_1, N_2$ s.t $L(N_1) = L(r_1)$ and $L(N_2) = L(r_2)$. We have already seen that there is a NFA $N$ s.t $L(N) = L(N_1) \cup L(N_2)$, hence $L(N) = L(r) = r = r_1 + r_2$. Use closure of NFA languages under concatenation and use closure of NFA languages under Kleene star.
For every regular expression $r$ show that there is a NFA $N$ such that $L(r) = L(N)$

Induction on length of $r$

Inductive cases:

- $r_1$, $r_2$ regular expressions and $r = r_1 + r_2$.
  By induction there are NFAs $N_1$, $N_2$ s.t $L(N_1) = L(r_1)$ and $L(N_2) = L(r_2)$. 
For every regular expression \( r \) show that there is a NFA \( N \) such that \( L(r) = L(N) \).

Induction on length of \( r \)

**Inductive cases:**

- \( r_1, r_2 \) regular expressions and \( r = r_1 + r_2 \). By induction there are NFAs \( N_1, N_2 \) s.t \( L(N_1) = L(r_1) \) and \( L(N_2) = L(r_2) \). We have already seen that there is NFA \( N \) s.t \( L(N) = L(N_1) \cup L(N_2) \), hence \( L(N) = L(r) \).
NFA and Regular Language

- For every regular expression $r$ show that there is a NFA $N$ such that $L(r) = L(N)$
- Induction on length of $r$

**Inductive cases:**

- $r_1, r_2$ regular expressions and $r = r_1 + r_2$.
  By induction there are NFA $N_1, N_2$ s.t $L(N_1) = L(r_1)$ and $L(N_2) = L(r_2)$. We have already seen that there is NFA $N$ s.t $L(N) = L(N_1) \cup L(N_2)$, hence $L(N) = L(r)$
- $r = r_1 \cdot r_2$. 
For every regular expression $r$ show that there is a NFA $N$ such that $L(r) = L(N)$

Induction on length of $r$

Inductive cases:

- $r_1, r_2$ regular expressions and $r = r_1 + r_2$.
  By induction there are NFA$s$ $N_1, N_2$ s.t $L(N_1) = L(r_1)$ and $L(N_2) = L(r_2)$. We have already seen that there is NFA $N$ s.t $L(N) = L(N_1) \cup L(N_2)$, hence $L(N) = L(r)$

- $r = r_1 \cdot r_2$. Use closure of NFA languages under concatenation
For every regular expression $r$ show that there is a NFA $N$ such that $L(r) = L(N)$. 

Induction on length of $r$ 

Inductive cases: 

- $r_1, r_2$ regular expressions and $r = r_1 + r_2$. By induction there are NFA$s N_1, N_2$ s.t $L(N_1) = L(r_1)$ and $L(N_2) = L(r_2)$. We have already seen that there is NFA $N$ s.t $L(N) = L(N_1) \cup L(N_2)$, hence $L(N) = L(r)$. 

- $r = r_1 \cdot r_2$. Use closure of NFA languages under concatenation 

- $r = (r_1)^*$. 

Use closure of NFA languages under Kleene star
For every regular expression $r$ show that there is a NFA $N$ such that $L(r) = L(N)$

Induction on length of $r$

Inductive cases:

- $r_1$, $r_2$ regular expressions and $r = r_1 + r_2$.

  By induction there are NFAs $N_1$, $N_2$ s.t
  
  $L(N_1) = L(r_1)$ and $L(N_2) = L(r_2)$. We have already seen that there is NFA $N$ s.t $L(N) = L(N_1) \cup L(N_2)$, hence $L(N) = L(r)$

- $r = r_1 \cdot r_2$. Use closure of NFA languages under concatenation

- $r = (r_1)^*$. Use closure of NFA languages under Kleene star
Example

\[(\varepsilon+0)(1+10)^*\]

\[
(\varepsilon+0) \rightarrow (1+10)^*
\]
Example

\[(\varepsilon + 0)(1+10)^*\]
Example

\[ (1+10) \ast \]

\[ \varepsilon \]

\[ 0 \]
Example

\[
\varepsilon 0 \quad (1+10) *
\]

\[
\varepsilon 0 \quad 10 1 *
\]
Example

\[ \varepsilon \]

\[ 0 \]

\[ (1+10) \]

\[ * \]

\[ 10 \]

\[ 1 \]

\[ 0 \]

\[ 1 \]
Final **NFA** simplified slightly to reduce states