Final  Mon May 6, 1:30 p - 4:30 p  
(175 min)

cheat sheet as before

7 problems

Undecidability

#11.  \( L = \{ <M> \mid \text{TM } M \text{ accepts exactly one string of length } n \geq 0 \} \)

\[ = \{ <M> \mid L(M) \text{ has exactly one string of } \cdots \} \]

is undecidable.

PT: Assume \( L \) is decided by algm \( \text{Funny}(<M>) \).
We'll design an algm to solve \( \text{Halting} \):

\( \text{Halting}(<M, w>) \):
1. encode the following TM \( M_w' \):

\[ M_w'(x): \begin{cases} \text{run } M \text{ on } w \\ \text{if } x \in 0^* \text{ return true} \\ \text{else } \text{false} \end{cases} \]

2. if \( \text{Funny}(<M_w>) \) return true 
else return false.

Then \( L(M_w') = \{ 0^* \text{ if } M \text{ halts on } w \} \)

\( 0^* \text{ else} \)
then \( L(M_w) = \{ 0 \uparrow \} \) if \( w \) is a valid input for \( M \)
else

\[ \text{Halting}(\langle M, w \rangle) \text{ returns true} \]
\[ \iff \text{Funny}(\langle M, w \rangle) \text{ returns true} \]
\[ \iff L(M_w) \text{ contains exactly 1 string of each length} \]
\[ \iff M \text{ halts on } w. \]

\[ \text{Halting is decidable: contra!} \quad \square \]

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**NP-Complete**

#3p. **Ultra-HC**:

Input: undirected graph \( G=(V,E) \).
Output: yes if \( \exists \) closed walk that visits every vertex exactly once, except \( \leq 1 \) vertex may be visited more than once

1. **Ultra-HC \( \in \) NP**:
   - Certificate: closed walk
   - Certifier: check \( C \) is ultra-Ham cycle polytime

2. **HC \( \leq_p \) Ultra-HC**:
   - Given input to HC: undirected graph \( G=(V,E) \),
   - Construct input to Ultra-HC: new graph \( G'=(V',E') \)
   - as follows:

   (CORRECTED SOL’N!!)

   ![Graph Diagram]
Fix vertex \( s \in V \), with neighbors \( u_1, \ldots, u_k \). Let \( G' \) be the graph:

\[
V' = V - \{s\} \cup \{s', s'', x, y\}.
\]

\[
E' = E - \{su_i : i = 1, \ldots, k\} \cup \{s' u_i : i = 1, \ldots, k\} \cup \{s'' u_i : i = 1, \ldots, k\} \cup \{s' x, x s'', x y\}
\]

Construction \( G \to G' \) takes polynomial time.

**Correctness:** \( \exists \) Hamilton cycle \( C \) in \( G \)

\[ \iff \exists \text{ ultra Hamilton cycle } C' \text{ in } G' \]

**Proof:** (\( \Rightarrow \)) Given \( C = v_1, v_2, \ldots, v_n, v_1 \), wlog say \( u_1 = s \).
Let \( C' = s' v_2, \ldots, v_n, s'', x, y, x, s' \), which is an ultra Hamilton cycle (only \( x \) is repeated).

(\( \Leftarrow \)) Given ultra Hamilton cycle \( C' \),
- \( x \) must be repeated
  (since it's the only way to get to \( y \))
- So all other vertices are visited once (including \( s', s'' \))
- So \( C' \) contains a path from \( s' \) to \( s'' \) visiting all vertices in \( V - \{s\} \) once
- This maps back to a Hamilton cycle in \( G \).
(Note: you may assume NP-completeness of 3SAT, indep set, vertex cover, clique, 3-coloring, Ham cycle/path, subset sum. Remember definitions of these problems.)

Greedy

#19 (mid2 ridden)

Given set of n intervals $[a_1, b_1], \ldots, [a_n, b_n]$, find smallest set $P$ of pts that stabs $X$. 
\[ P = \emptyset \]

\[
\text{repeat } \{
\begin{align*}
\text{pick } i \text{ with smallest } b_i \\
\text{insert } b_i \text{ to } P \\
\text{remove all intervals stabbed by } b_i
\end{align*}
\}
\]

**Correctness Pf:**

Let \( P^* \) be opt soln.

Let \( i \) minimizes \( b_i \).

Let \( x^* \in P^* \) that stabs \( (a_i, b_i) \)

\[ a_i \quad b_i \]

\[ x^* \]

For every interval \( (a_j, b_j) \) stabbed by \( x^* \),

\[ a_i \quad b_i \]

\[ x^* \]

Know \( b_i \leq b_j \).

\[ \Rightarrow \] \( (a_j, b_j) \) is stabbed by \( b_i \).

\[ \Rightarrow \] \( P^* - \{x^*\} \cup \{b_i\} \) is a feasible soln.
$P - \{x^i\} \cup \{b_i\}$ is a tension join & is also optimal & uses $b_i$.

Repeat arg.

$\Rightarrow \exists$ opt soln that agrees with greedy.  \qed