Last Time:
- SAT is NP-complete.
- 3SAT is NP-complete.

Recipe: To show L is NP-complete:
① L ∈ NP
② L₀ ≤ₚ L for some known NP-complete L₀.

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**Independent Set**

Input: G=(V,E), integer k
Output: yes iff ∃ independent set S of size > k.

Thus, **Independent Set is NP-complete.**

Pf: ① Independent ∈ NP:
Certificate: subset S ⊆ V.
Certificate checks |S| ≤ k
& ∀ u, v ∈ S, u ≠ v ∈ E, poly-size

② 3SAT ≤ₚ Independent-Set:
Given input to 3SAT: a 3CNF formula F
with n vars, m clauses,
construct input to Independent-Set: graph G=(V,E),
& integer k
Construct input to Indep Set: graph \( u-vz \), & integer \( k \).

as follows:

for each clause \( \alpha_{ij} \lor \alpha_{i2} \lor \alpha_{i3} \),
create 3 vertices \( v_{i1}, v_{i2}, v_{i3} \)

& 3 edges \( v_{i1} v_{i2}, v_{i2} v_{i3}, v_{i3} v_{i1} \)

whenever \( \alpha_{ij} = \overline{\alpha_{ij}} \),
add edge \( v_{ij} v_{ij} \) — "cross" edges

Set \( k = m \).
Construction \( F \rightarrow (G, k) \) takes poly time. 
\( (O(m) \) vertices; \( G \), \( O(m^2) \) edges).

E.g. given \( F = (x_1 \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor x_4) \land (\overline{x_2} \lor x_3 \lor x_4) \land (x_1 \lor \overline{x_3} \lor x_4) \) 
\((x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 0) \)

Construct \( G : \)

Correctness: \( \exists \) assignment that makes \( F \) true
\( \iff \exists \) indep set \( S \) for \( G \) of size \( \geq k \).

Pf: \((\Rightarrow)\) Let \( A \) be a sat. assignment for \( F \).
construct a subset \( S \) as follows:
Define a subset $S$ as follows:
for each clause $\alpha_{ij} \lor \alpha_{i'j} \lor \alpha_{i''j}$, pick some $j$ s.t. $\alpha_{ij}$ is true.
put $v_{ij}$ in $S$.

Then $|S| = m = k$.
$S$ is an independent set.
(check triangle edges ✓
   cross edge ✓)

$(\Leftarrow)$ Let $S$ be an independent set of $G$ of size $\geq k$.

Define an assignment $\alpha$ as follows:
whenever $v_{ij}$ is in $S$,
set $\alpha_{ij}$ to true.
(Set all remaining vars arbitrarily.)

Then $\alpha$ is consistent:
if $\alpha_{ij} = \overline{\alpha_{ij}}$, can't have $v_{ij}, v_{i'j}$ both in $S$ because of cross edge
so can't set both $\alpha_{ij}, \alpha_{i'j}$ true.

$\alpha$ is satisfying:
for each triangle,
at most one $v_{ij}$ is in $S$.

but since $|S| \geq k = m$,
exactly one $v_{ij}$ is in $S$
so $\alpha_{ij} \lor \alpha_{i'j} \lor \alpha_{i''j}$ is true for $\alpha_i$. $\square$
Cor

Vertex-Cover is NP-complete.
Set-Cover " "

\underline{Hamiltonian Cycle (HC)}

Input: graph \( G = (V, E) \)
Output: yes iff \( \exists \) cycle visiting every vertex exactly once

\text{e.g.}

\begin{itemize}
  \item \text{yes}
  \item \text{no}
\end{itemize}

\text{variants:}

\begin{itemize}
  \item \text{dir.}
  \item \text{Ham path}
\end{itemize}

\text{Note: dir-HC \leq_\text{P} undir-HC}

\text{Thm (Karp'72)} HC is NP-complete.
Pf:  1. HC ∈ NP
2. Vertex-Cover ≤p dir-HC:

Given input to VertexCover: undir graph \( G = (V, E) \), integer \( k \),

Construct input to dir-HC: dir graph \( G' \), as follows:

for each vertex \( v \in V \),
    draw a "line" \( e_v \)

\[ e_v \longrightarrow \]

for each edge \( uv \in E \),
    add a gadget between \( e_u \) & \( e_v \)

\[ e_u \longrightarrow \]
\[ e_v \longrightarrow \]

(To be cont'd)

\[ \begin{array}{c}
\text{e.g. } \quad G \quad \begin{array}{c}
A \quad \rightarrow \quad B \\
\quad \downarrow \quad \quad \quad \downarrow \\
C \quad \rightarrow \quad D \\
\end{array} \\
\end{array} \]

\[ k = 2 \quad \{B, C\} \]