Fall course announcements:
Sariel CS 498 Rand. Algs
me CS 598 Geometric Data Structures

PART III: Undecidability & NP-Completeness

how to prove problem is hard to solve...

**Def** L is decidable

if ∃ TM/program M that
halts on all input & accepts w if w ∈ L
rejects w if w ∉ L

**Turing's Thm**

\[
\text{TM-Halt} = \{ \langle M, w \rangle \mid \text{TM } M \text{ halts on input string } w \}
\]

is undecidable.

Similarly,
Similarly,
\[ \text{TM-Acc} = \{ <M,w> | \text{TM } M \text{ accepts input string } w \} \]
is undecidable.

**Pf:** By contradiction.

Assume \( \text{TM-Acc} \) is decidable, by TM/alg \( \text{Macc} \).

Will construct a counterex \( M_{bad}, w_{bad} \):

\( M_{bad} \) is this program:

- on input \( <M> \),
- run \( \text{Macc} \) on \( <M,<M>> \)
  - if \( \text{Macc} \) accepts, reject
  - else accept

\[ w_{bad} = <M_{bad}>. \]

**Case 1.** \( \text{Macc} \) accepts \( <M_{bad},<M_{bad}> > \).
\( M_{bad} \) rejects \( <M_{bad}> \) : wrong!

**Case 2.** \( \text{Macc} \) rejects \( <M_{bad},<M_{bad}> > \)
\( M_{bad} \) accepts : wrong!

\[ \square \]

From one hard problem, can prove other problems hard by reduction.

**Ex1.** \( \text{TM-Acc-All} = \{ <M> | \text{TM } M \text{ accepts all inputs} \} \)
\[ = \{ <M> | L(M) = \Sigma^* \} \]
is undecidable.
\[
\text{[appl: main (\text{int } n):}
\begin{align*}
&\text{if } n \text{ is even then accept} \\
&\text{if sum of all divisors of } n \text{ is } 2n \\
&\text{then reject} \\
&\text{else accept}
\end{align*}
\]

**Pf:** By contradiction.

Assume TM-Acc-All is decidable by algm \( M_{\text{all}} \).

Will give an algm \( M_{\text{acc}} \) to decide TM-Acc:

On input \(<M, w>\),

construct \(<M'_w>\) encoding of a new TM \( M'_w \),

where on input \( x \),

\( M'_w \) ignores \( x \) & just simulate \( M \) on \( w \).

i.e. given string \( M = "f(...) \{\ldots\}" \)

and string \( w \),

construct new string

\[ M'_w = "f(\ldots) \{\ldots\} \]

\[ \text{main( string } x) \}\}

\[ \text{return } f("w") \} \]

\]

String manipulation:
Linear time

\[ M_{\text{acc}} \]

\[ <M> \quad <M'_w> \quad M_{\text{all}} \quad \text{accept/reject} \]

\]
Then run $M_{all}$ on $\langle M', w \rangle$
accept iff $M_{all}$ accepts.

Correctness:
$M_{acc}$ accepts $\langle M, w \rangle$
$\iff M_{all}$ accepts $\langle M', w \rangle$
$\iff M'$ accepts all input
$\iff M$ accepts $w$.

$\Rightarrow$ $TM\text{-}Acc$ is decidable. Contradiction! \(\Box\)

**Ex2**
$TM\text{-}Acc\text{-}Same = \{ \langle M \rangle \mid L(M) \neq \emptyset \}$
is undecidable.

*Same proof as Ex1!*

$TM\text{-}Empty = \{ \langle M \rangle \mid L(M) = \emptyset \}$
is undecidable
(by closure under complement)

**Ex3**
$TM\text{-}EQUIV = \{ \langle M_1, M_2 \rangle \mid L(M_1) = L(M_2) \}$
is undecidable.

**Pf:** By contradiction.
Assume $TM\text{-}EQUIV$ is decidable, by algm $M_{equiv}$.
Will give algm to decide $TM\text{-}Acc\text{-}All$:

On input $\langle M \rangle$,
create $TM M_2$ with $L(M_2) = \Sigma^*$
just run $M_{equiv}$ on $\langle M, M_2 \rangle$. \_
Ex4  \[ \text{Tm-Reg} = \{ \langle M \rangle \mid L(M) \text{ is regular for TM } M \} \]
is undecidable.

**Pf:** Very similar to Ex1.

By contradiction.

Assume Tm-Reg is decidable, by alg'm Mreg.

Will give an alg'n Macc to decide Tm-Acc

as follows:

1. On input \( \langle M, w \rangle \),
   1. construct encoding \( \langle M_w' \rangle \) of
      a new TM \( M_w' \)
      where
      - on input \( x \),
      - if \( M \) accepts \( w \)
        then \( M_w' \) accepts \( x \)
        else reject
      - iff \( x \) is a palindrome

2. run \( M_{\text{reg}} \) on \( \langle M_w' \rangle \)

3. accept iff \( M_{\text{reg}} \) rejects

**Correctness:**

\[ L(M_w') = \begin{cases} \{ x \mid x \text{ is a palindrome} \} & \text{if } M \text{ accepts } w \\ \emptyset & \text{if } M \text{ rejects } w \\ \text{a reg. lang.} & \text{or } M \text{ does not halt on } w \end{cases} \]
Macc accepts \( \langle M, w \rangle \)
\[\iff\] Mreg rejects \( \langle M_w \rangle \)
\[\iff\] \( L(M_w) \) is not regular
\[\iff\] \( M \) accepts \( w \).

So, TM-Acc is decidable: Contradiction! \( \Box \)

Remark: Proof very general

**Rice's Thm** Let \( P \) be a property about langs.
\[\{ \langle M \rangle \mid L(M) \text{ has property } P \}\]
is undecidable

if \( P \) is nontrivial

(i.e. some decidable lang. has \( P \) & some """" does not have \( P \))

**Exs** \[\{ \langle M \rangle \mid M \text{ accepts exactly 374 strings} \}\]
undecidable by Rice

\[\{ \langle M \rangle \mid M \text{ runs in } \leq 100 \text{ steps} \}\]
Rice not applicable!

Other undecidable problems

a) \[\{ \langle G \rangle \mid L(G) = \Sigma^* \text{ for CFG } G \}\]

b) **Hilbert’s 10th Problem:**
b) Hilbert’s 10th Problem:
   does a polynomial eq’n has integer solns?
   \[ x^7 + y^7 = z^7 + 3 \]

c) tiling

\[ \begin{array}{c}
\includegraphics{image1} \\
\includegraphics{image2}
\end{array} \]