Last Time:

**DFA**
- $O(1)$ memory
- read input from L to R

Thus if $L_1$ accepted by DFA $M_1$,
$L_2$ accepted by DFA $M_2$,
$L_1 \cap L_2 \subseteq \ldots$ by some DFA.
$L_1 \subseteq \ldots \subseteq \ldots$

Cor can also do $L_1 \cup L_2$
& $L_1 \setminus L_2$
(because $L_1 \cup L_2 = \overline{L_1 \cap L_2}$).

Will prove DFA $\leftrightarrow$ regular langs.

**Non-deterministic Finite Automata (NFA)**

- allow choices
- allow $\epsilon$-transitions

**Ex:** all strings ending with 01

![Diagram of NFA](https://via.placeholder.com/150)

not valid DFA e.g. $\delta(q_0,0) = q_0$ or $q_4$?
but ok for NFA $\delta(q_1,0)$ is undef.
accept iff $\exists$ path leading to accept state

\[ \text{e.g. } \begin{array}{l}
001 \rightarrow q_0 \rightarrow q_1 \rightarrow q_2 
\end{array} \]

reject iff $\forall$ path, lead to non-accept state

not realistic machine!
( not obviously convertible to efficient program )

$\exists \epsilon$

\[ 0^* (01)^* 1^* \]

\[ \begin{array}{ccc}
q_0 & \rightarrow & q_1 \\
\epsilon & \rightarrow & q_2 \\
q_1 & \rightarrow & q_3 \\
\epsilon & \rightarrow & q_4 \\
\end{array} \]

( $\epsilon$-transitions don’t consume input )

**Formal Def**

An NFA is $M=(Q, \Sigma, \delta, s, A)$ like before,

except

(before $s: Q \times \Sigma \rightarrow Q$)

$\delta : Q \times (\Sigma \cup \{\epsilon\}) \rightarrow \mathcal{P}(Q)$

power set of $Q$

= $\{\text{all subsets of } Q\}$
$\delta(q_0,0) = \{q_0, q_1\}$
$\delta(q_1,1) = \emptyset$

$\delta(q_0,3) = \{q_1\}$

**Def** Given $q \in Q$, define $\varepsilon$-reach($q$) inductively:

(i) $q$ is in $\varepsilon$-reach($q$)

(ii) if $q'$ is in $\varepsilon$-reach($q$),
    & $q'' \in \delta(q',3)$,
    - then $q''$ is in $\varepsilon$-reach($q$).
    (Nothing else is in).

$\varepsilon$-reach($q_0$) = $\{q_0, q_1, q_3\}$
$\varepsilon$-reach($q_1$) = $\{q_1, q_3\}$

**Def** Define extended transition fn $\delta^* : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$ inductively:

(i) $\delta^*(q,3) = \varepsilon$-reach($q$)

(ii) $\delta^*(q,x) = \bigcup_{q' \in \varepsilon$-reach($q$)} \bigcup_{q'' \in \delta(q',x)} \delta^*(q'',y)$
    for $x = ay$  \(a \in \Sigma, y \in \Sigma^*)\)

Define $L(M) = \{x \in \Sigma^* | q_0 \in \delta^*(q_0, x)\}$
Define \( L(M) = \{ w \in \{0, 1\}^* \mid \text{lang\ accepted by } M \text{ \ by } \delta(s, x) \cap A \neq \emptyset \} \)

**Ex. a)** \((\varepsilon + 0)(01 + 001)^*(10)^*\)

b) all strings whose 5th symbol from right is a 0.

\(\text{--0--xx} \quad \text{(any DFA needs 32 states!)}\)

c) all strings not ending with 01

\(\text{WRONG}\)
Thm: If $L_1$ is accepted by NFA $M_1$, $L_2$ is accepted by NFA $M_2$, then (i) $L_1 \cup L_2$ is accepted by some NFA. (ii) $L_1 \cdot L_2$ is accepted by some NFA. (iii) $L_1^*$ is accepted by some NFA.

Pf: Given $M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1)$, $M_2 = (Q_2, \Sigma, \delta_2, s_2, A_2)$

(i) **Union**

(ii) **Concat**

(iii) **Star**