Describe recursive backtracking algorithms for the following problems. Don’t worry about running times.

1. Given an array $A[1..n]$ of integers, compute the length of a longest increasing subsequence.

**Solution:**

[#1 of $\infty$] Add a sentinel value $A[0] = -\infty$. Let $LIS(i, j)$ denote the length of the longest increasing subsequence of $A[j..n]$ where every element is larger than $A[i]$. This function obeys the following recurrence:

$$LIS(i, j) = \begin{cases} 
0 & \text{if } j > n \\
LIS(i, j + 1) & \text{if } j \leq n \text{ and } A[i] \leq A[j] \\
\max \{LIS(i, j + 1), 1 + LIS(j, j + 1)\} & \text{otherwise}
\end{cases}$$

We need to compute $LIS(0, 1)$.

**Solution:**

[#2 of $\infty$] Add a sentinel value $A[n + 1] = -\infty$. Let $LIS(i, j)$ denote the length of the longest increasing subsequence of $A[1..j]$ where every element is smaller than $A[j]$. This function obeys the following recurrence:

$$LIS(i, j) = \begin{cases} 
0 & \text{if } i < 1 \\
LIS(i - 1, j) & \text{if } i \geq 1 \text{ and } A[i] \leq A[j] \\
\max \{LIS(i - 1, j), 1 + LIS(i - 1, i)\} & \text{otherwise}
\end{cases}$$

We need to compute $LIS(n, n + 1)$.

**Solution:**

[#3 of $\infty$] Let $LIS(i)$ denote the length of the longest increasing subsequence of $A[i..n]$ that begins with $A[i]$. This function obeys the following recurrence:

$$LIS(i) = \begin{cases} 
1 & \text{if } A[j] \leq A[i] \text{ for all } j > i \\
1 + \max \{LIS(j)\} & j > i \text{ and } A[j] > A[i] \text{ otherwise}
\end{cases}$$

(The first case is actually redundant if we define $\max \emptyset = 0$.) We need to compute $\max_i LIS(i)$.

**Solution:**

[#4 of $\infty$] Add a sentinel value $A[0] = -\infty$. Let $LIS(i)$ denote the length of the longest increasing subsequence of $A[i..n]$ that begins with $A[i]$. This function obeys the following recurrence:

$$LIS(i) = \begin{cases} 
1 & \text{if } A[j] \leq A[i] \text{ for all } j > i \\
1 + \max \{LIS(j)\} & j > i \text{ and } A[j] > A[i] \text{ otherwise}
\end{cases}$$

(The first case is actually redundant if we define $\max \emptyset = 0$.) We need to compute $LIS(0) - 1$; the $-1$ removes the sentinel $-\infty$ from the start of the subsequence.
Solution:
[#5 of ∞] Add sentinel values $A[0] = -\infty$ and $A[n + 1] = \infty$. Let $LIS(j)$ denote the length of the longest increasing subsequence of $A[1..j]$ that ends with $A[j]$. This function obeys the following recurrence:

$$LIS(j) = \begin{cases} 1 & \text{if } j = 0 \\ 1 + \max \{LIS(i) \mid i < j \text{ and } A[i] < A[j]\} & \text{otherwise} \end{cases}$$

We need to compute $LIS(n + 1) - 2$; the $-2$ removes the sentinels $-\infty$ and $\infty$ from the subsequence.

2 Given an array $A[1..n]$ of integers, compute the length of a longest decreasing subsequence.

Solution:
[one of many] Add a sentinel value $A[0] = \infty$. Let $LDS(i, j)$ denote the length of the longest decreasing subsequence of $A[j..n]$ where every element is smaller than $A[i]$. This function obeys the following recurrence:

$$LDS(i, j) = \begin{cases} 0 & \text{if } j > n \\ LDS(i, j + 1) & \text{if } j \leq n \text{ and } A[i] \leq A[j] \\ \max \{LDS(i, j + 1), 1 + LIS(j, j + 1)\} & \text{otherwise} \end{cases}$$

We need to compute $LDS(0, 1)$.

Solution:
[clever] Multiply every element of $A$ by $-1$, and then compute the length of the longest increasing subsequence using the algorithm from problem 1.

3 Given an array $A[1..n]$ of integers, compute the length of a longest alternating subsequence.

Solution:
[one of many] We define two functions:

- Let $LAS^+(i, j)$ denote the length of the longest alternating subsequence of $A[j..n]$ whose first element (if any) is larger than $A[i]$ and whose second element (if any) is smaller than its first.
- Let $LAS^-(i, j)$ denote the length of the longest alternating subsequence of $A[j..n]$ whose first element (if any) is smaller than $A[i]$ and whose second element (if any) is larger than its first.

These two functions satisfy the following mutual recurrences:

$$LAS^+(i, j) = \begin{cases} 0 & \text{if } j > n \\ LAS^+(i, j + 1) & \text{if } j \leq n \text{ and } A[i] \leq A[j] \\ \max \{LAS^+(i, j + 1), 1 + LAS^-(j, j + 1)\} & \text{otherwise} \end{cases}$$

$$LAS^-(i, j) = \begin{cases} 0 & \text{if } j > n \\ LAS^-(i, j + 1) & \text{if } j \leq n \text{ and } A[i] \geq A[j] \\ \max \{LAS^-(i, j + 1), 1 + LAS^+(j, j + 1)\} & \text{otherwise} \end{cases}$$

To simplify computation, we consider two different sentinel values $A[0]$. First we set $A[0] = -\infty$ and let $\ell^+ = LAS^+(0, 1)$. Then we set $A[0] = +\infty$ and let $\ell^- = LAS^-(0, 1)$. Finally, the length of the longest alternating subsequence of $A$ is $\max\{\ell^+, \ell^-\}$.
Solution:
[one of many] We define two functions:

- Let $LAS^+(i)$ denote the length of the longest alternating subsequence of $A[i \ldots n]$ that starts with $A[i]$ and whose second element (if any) is larger than $A[i]$.
- Let $LAS^-(i)$ denote the length of the longest alternating subsequence of $A[i \ldots n]$ that starts with $A[i]$ and whose second element (if any) is smaller than $A[i]$.

These two functions satisfy the following mutual recurrences:

\[
LAS^+(i) = \begin{cases} 
1 & \text{if } A[j] \leq A[i] \text{ for all } j > i \\
1 + \max \{LAS^-(j)\} & \text{if } A[j] > A[i] \text{ for all } j > i 
\end{cases}
\]

\[
LAS^-(i) = \begin{cases} 
1 & \text{if } A[j] \geq A[i] \text{ for all } j > i \\
1 + \max \{LAS^+(j)\} & \text{if } A[j] < A[i] \text{ for all } j > i 
\end{cases}
\]

We need to compute $\max_i \max_j \{LAS^+(i), LAS^-(i)\}$.

To think about later:

1. Given an array $A[1 \ldots n]$ of integers, compute the length of a longest convex subsequence of $A$.

Solution:

Let $LCS(i, j)$ denote the length of the longest convex subsequence of $A[i \ldots n]$ whose first two elements are $A[i]$ and $A[j]$. This function obeys the following recurrence:

\[
LCS(i, j) = 1 + \max \{LCS(j, k)\} \quad j < k \leq n \text{ and } A[i] + A[k] > 2A[j]
\]

Here we define $\max \emptyset = 0$; this gives us a working base case. The length of the longest convex subsequence is $\max_{1 \leq i < j \leq n} LCS(i, j)$.

Solution:

[with sentinels] Assume without loss of generality that $A[i] \geq 0$ for all $i$. (Otherwise, we can add $|m|$ to each $A[i]$, where $m$ is the smallest element of $A[1 \ldots n]$.) Add two sentinel values $A[0] = 2M + 1$ and $A[-1] = 4M + 3$, where $M$ is the largest element of $A[1 \ldots n]$.

Let $LCS(i, j)$ denote the length of the longest convex subsequence of $A[i \ldots n]$ whose first two elements are $A[i]$ and $A[j]$. This function obeys the following recurrence:

\[
LCS(i, j) = 1 + \max \{LCS(j, k)\} \quad j < k \leq n \text{ and } A[i] + A[k] > 2A[j]
\]

Here we define $\max \emptyset = 0$; this gives us a working base case.

Finally, we claim that the length of the longest convex subsequence of $A[1 \ldots n]$ is $LCS(-1, 0) - 2$.


On the other hand, removing $A[-1]$ and $A[0]$ from any convex subsequence of $A[-1 \ldots n]$ leaves a convex subsequence of $A[1 \ldots n]$. So the longest subsequence of $A[1 \ldots n]$ has length at least $LCS(-1, 0) - 2$. ■
Given an array $A[1..n]$, compute the length of a longest palindrome subsequence of $A$.

**Solution:**

[naive] Let $LPS(i, j)$ denote the length of the longest palindrome subsequence of $A[i..j]$. This function obeys the following recurrence:

$$LPS(i, j) = \begin{cases} 
0 & \text{if } i > j \\
1 & \text{if } i = j \\
\max \left\{ LPS(i+1, j), LPS(i, j-1) \right\} & \text{if } i < j \text{ and } A[i] \neq A[j] \\
2 + LPS(i+1, j-1) & \text{otherwise}
\end{cases}$$

We need to compute $LPS(1, n)$.

**Solution:**

[with greedy optimization] Let $LPS(i, j)$ denote the length of the longest palindrome subsequence of $A[i..j]$. Before stating a recurrence for this function, we make the following useful observation.\(^1\)

**Claim 0.1.** If $i < j$ and $A[i] = A[j]$, then $LPS(i, j) = 2 + LPS(i+1, j-1)$.

**Proof:** Suppose $i < j$ and $A[i] = A[j]$. Fix an arbitrary longest palindrome subsequence $S$ of $A[i..j]$. There are four cases to consider.

- If $S$ uses neither $A[i]$ nor $A[j]$, then $A[i] \bullet S \bullet A[j]$ is a palindrome subsequence of $A[i..j]$ that is longer than $S$, which is impossible.
- Finally, $S$ might include both $A[i]$ and $A[j]$.

In all cases, we find either a contradiction or a longest palindrome subsequence of $A[i..j]$ that uses both $A[i]$ and $A[j]$.

Claim 1 implies that the function $LPS$ satisfies the following recurrence:

$$LPS(i, j) = \begin{cases} 
0 & \text{if } i > j \\
1 & \text{if } i = j \\
\max \{ LPS(i+1, j), LPS(i, j-1) \} & \text{if } i < j \text{ and } A[i] \neq A[j] \\
2 + LPS(i+1, j-1) & \text{otherwise}
\end{cases}$$

We need to compute $LPS(1, n)$.