Give context-free grammars for each of the following languages.

1. \( \{0^{2n}1^n \mid n \geq 0\} \)

   **Solution:** \( S \rightarrow \varepsilon \mid 00S1 \).

2. \( \{0^m1^n \mid m \neq 2n\} \)

   (Hint: If \( m \neq 2n \), then either \( m < 2n \) or \( m > 2n \).)

   **Solution:**

   To simplify notation, let \( \Delta(w) = \#(0,w) - 2\#(1,w) \). Our solution follows the following logic. Let \( w \) be an arbitrary string in this language.

   - Because \( \Delta(w) \neq 0 \), then either \( \Delta(w) > 0 \) or \( \Delta(w) < 0 \).
   - If \( \Delta(w) > 0 \), then \( w = 0^i z \) for some integer \( i > 0 \) and some suffix \( z \) with \( \Delta(z) = 0 \).
   - If \( \Delta(w) < 0 \), then \( w = x1^j \) for some integer \( j > 0 \) and some prefix \( x \) with either \( \Delta(x) = 0 \) or \( \Delta(x) = 1 \).
   - Substrings with \( \Delta = 0 \) is generated by the previous grammar; we need only a small tweak to generate substrings with \( \Delta = 1 \).

   Here is one way to encode this case analysis as a CFG. The nonterminals \( M \) and \( L \) generate all strings where the number of 0s is More or Less than twice the number of 1s, respectively. The last nonterminal generates strings with \( \Delta = 0 \) or \( \Delta = 1 \).

   \[
   S \rightarrow M \mid L \quad \{0^m1^n \mid m \neq 2n\} \\
   M \rightarrow 0M \mid 0E \quad \{0^m1^n \mid m > 2n\} \\
   L \rightarrow L1 \mid E1 \quad \{0^m1^n \mid m < 2n\} \\
   E \rightarrow \varepsilon \mid 0 \mid 00E1 \quad \{0^m1^n \mid m = 2n \text{ or } 2n + 1\}
   \]

   Here is a different correct solution using the same logic. We either identify a non-empty prefix of 0s or a non-empty prefix of 1s, so that the rest of the string is as “balanced” as possible. We also generate strings with \( \Delta = 1 \) using a separate non-terminal.

   \[
   S \rightarrow AE \mid EB \mid FB \quad \{0^m1^n \mid m \neq 2n\} \\
   A \rightarrow 0 \mid 0A \\
   B \rightarrow 1 \mid 1B \\
   E \rightarrow \varepsilon \mid 00E1 \\
   F \rightarrow 0E \\
   \]

   \[
   0^+ = \{0^i \mid i \geq 1\} \\
   1^+ = \{1^j \mid j \geq 1\} \\
   \]

   \[
   \{0^m1^n \mid m = 2n\} \\
   \{0^m1^n \mid m = 2n + 1\}
   \]

   Alternatively, we can separately generate all strings of the form \( 0^\text{odd}1^* \), so that we don’t have to worry about the case \( \Delta = 1 \) separately.

   \[
   S \rightarrow D \mid M \mid L \\
   D \rightarrow 0 \mid 00D \mid D1 \\
   M \rightarrow 0M \mid 0E \\
   L \rightarrow L1 \mid E1 \\
   E \rightarrow \varepsilon \mid 00E1 \\
   \]

   \[
   \{0^m1^n \mid m \neq 2n\} \\
   \{0^m1^n \mid m \text{ is odd}\} \\
   \{0^m1^n \mid m > 2n\} \\
   \{0^m1^n \mid m < 2n \text{ and } m \text{ is even}\} \\
   \{0^m1^n \mid m = 2n\}
   \]
Solution:
Intuitively, we can parse any string $w \in L$ as follows. First, remove the first $2k$ 0s and the last $k$ 1s, for the largest possible value of $k$. The remaining string cannot be empty, and it must consist entirely of 0s, entirely of 1s, or a single 0 followed by 1s.

$$
S \rightarrow 00S | A | B | C \quad \quad \{0^n1^n | m \neq 2n\}
$$

$$
A \rightarrow 0 \mid 0A \quad 0^+ \\
B \rightarrow 1 \mid 1B \quad 1^+ \\
C \rightarrow 0 \mid 0B \quad 01^+
$$

Let’s elaborate on the above, since $k$ is maximal, $w = 0^{2k}w'1^k$. If $w'$ starts with 00, and ends with a 1, then we can increase $k$ by one. As such, $w'$ is either in $0^+$ or $1^+$. If $w'$ contains both 0s and 1s, then it can contain only a single 0, followed potentially by $1^+$. We conclude that $w' \in 0^+ + 1^+ + 01^+$.

3. $\{0, 1\}^* \setminus \{0^{2n}1^n | n \geq 0\}$

Solution:
This language is the union of the previous language and the complement of $0^*1^*$, which is $(0+1)^*10(0+1)^*$.

$$
S \rightarrow T | X \quad \quad \quad \quad \{0, 1\}^* \setminus \{0^{2n}1^n | n \geq 0\}
$$

$$
T \rightarrow 00T1 | A | B | C \quad \quad \{0^n1^n | m \neq 2n\}
$$

$$
A \rightarrow 0 \mid 0A \quad 0^+ \\
B \rightarrow 1 \mid 1B \quad 1^+ \\
C \rightarrow 0 \mid 0B \quad 01^+
$$

$$
X \rightarrow Z10Z \quad (0 + 1)^*10(0 + 1)^* \\
Z \rightarrow \varepsilon \mid 0Z \mid 1Z \quad (0 + 1)^*
$$

Work on these later:
4. $\{w \in \{0, 1\}^* | \#(0, w) = 2 \cdot \#(1, w)\}$ – Binary strings where the number of 0s is exactly twice the number of 1s.

Solution:
$$
S \rightarrow \varepsilon | SS | 00S1 | 0S1S0 | 1S00.
$$

Here is a sketch of a correctness proof; a more detailed proof appears in the homework.

For any string $w$, let $\Delta(w) = \#(0, w) - 2 \cdot \#(1, w)$. Suppose $w$ is a binary string such that $\Delta(w) = 0$. Suppose $w$ is nonempty and has no non-empty proper prefix $x$ such that $\Delta(x) = 0$. There are three possibilities to consider:

- Suppose $\Delta(x) > 0$ for every proper prefix $x$ of $w$. In this case, $w$ must start with 00 and end with 1. Thus, $w = 00x1$ for some string $x \in L$. 

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- Suppose $\Delta(x) < 0$ for every proper prefix $x$ of $w$. In this case, $w$ must start with 1 and end with 00. Let $x$ be the shortest non-empty prefix with $\Delta(x) = 1$. Thus, $w = 1X00$ for some string $x \in L$.

- Finally, suppose $\Delta(x) > 0$ for some prefix $x$ and $\Delta(x') < 0$ for some longer proper prefix $x'$. Let $x'$ be the shortest non-empty proper prefix of $w$ with $\Delta < 0$. Then $x' = 0y1$ for some substring $y$ with $\Delta(y) = 0$, and thus $w = 0y1z0$ for some strings $y, z \in L$.

\[
\{0, 1\}^* \setminus \{ww \mid w \in \{0, 1\}^*\}.
\]

**Solution:**

All strings of odd length are in $L$.

Let $w$ be any even-length string in $L$, and let $m = |w|/2$. For some index $i \leq m$, we have $w_i \neq w_{m+i}$. Thus, $w$ can be written as either $x1y0z$ or $x0y1z$ for some substrings $x, y, z$ such that $|x| = i - 1$, $|y| = m - 1$, and $|z| = m - i$. We can further decompose $y$ into a prefix of length $i - 1$ and a suffix of length $m - i$. So we can write any even-length string $w \in L$ as either $x1x'z'0z$ or $x0x'z'1z$, for some strings $x, x', z, z'$ with $|x| = |x'| = i - 1$ and $|z| = |z'| = m - i$. Said more simply, we can divide $w$ into two odd-length strings, one with a 0 at its center, and the other with a 1 at its center.

\[
S \rightarrow AB \mid BA \mid A \mid B \quad \text{strings not of the form } ww
\]
\[
A \rightarrow 0 \mid \Sigma \quad \text{odd-length strings with 0 at center}
\]
\[
B \rightarrow 1 \mid \Sigma \quad \text{odd-length strings with 1 at center}
\]
\[
\Sigma \rightarrow 0 \mid 1 \quad \text{single character}
\]