Prove that each of the following languages is \textbf{not} regular.

\begin{enumerate}
\item \{0^{2^n} \mid n \geq 0\}
\end{enumerate}

\textbf{Solution:}

Let \( F = L = \{0^{2^n} \mid n \geq 0\} \).

Let \( x \) and \( y \) be arbitrary elements of \( F \).

Then \( x = 0^{2i} \) and \( y = 0^{2j} \) for some non-negative integers \( x \) and \( y \).

Let \( z = 0^{2i} \).

Then \( xz = 0^{2i}0^{2i} = 0^{2i+1} \in L \).

And \( yz = 0^{2j}0^{2i} = 0^{2i+2j} \notin L \), because \( i \neq j \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

\begin{enumerate}
\item \{0^{2^n}1^n \mid n \geq 0\}
\end{enumerate}

\textbf{Solution:}

For any non-negative integers \( i \neq j \), the strings \( 0^{2i} \) and \( 0^{2j} \) are distinguished by the suffix \( 0^{2i} \), because \( 0^{2i}0^{2i} = 0^{2i+1} \in L \) but \( 0^{2i}0^{2i} = 0^{2i+2j} \notin L \). Thus \( L \) itself is an infinite fooling set for \( L \).

\begin{enumerate}
\item \{0^{2n}1^n \mid n \geq 0\}
\end{enumerate}

\textbf{Solution:}

Let \( F \) be the language \( 0^* \).

Let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Let \( z = 0^{i+1} \).

Then \( xz = 0^{2i+1}i \in L \).

And \( yz = 0^{i+j+1}i \notin L \), because \( i + j \neq 2i \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

\textbf{Solution:}

For all non-negative integers \( i \neq j \), the strings \( 0^i \) and \( 0^j \) are distinguished by the suffix \( 0^i1^i \), because \( 0^{2i+1}i \in L \) but \( 0^{i+j}1^i \notin L \). Thus, the language \( 0^* \) is an infinite fooling set for \( L \).
Solution:

For all non-negative integers \( i \neq j \), the strings \( 0^{2i} \) and \( 0^{2j} \) are distinguished by the suffix \( 1^i \), because \( 0^{2i+1} \in L \) but \( 0^{2j+1} \notin L \). Thus, the language \((00)^*\) is an infinite fooling set for \( L \).

3 \( \{0^n1^n \mid m \neq 2n\} \)

Solution:

Let \( F \) be the language \( 0^* \).

Let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Let \( z = 0^i1^i \).

Then \( xz = 0^{2i+1} \notin L \).

And \( yz = 0^{i+j}1^i \in L \), because \( i + j \neq 2i \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

Solution:

For all non-negative integers \( i \neq j \), the strings \( 0^{2i} \) and \( 0^{2j} \) are distinguished by the suffix \( 1^i \), because \( 0^{2i+1} \notin L \) but \( 0^{2j+1} \in L \). Thus, the language \((00)^*\) is an infinite fooling set for \( L \).

4 Strings over \( \{0,1\} \) where the number of 0s is exactly twice the number of 1s.

Solution:

Let \( F \) be the language \( 0^* \).

Let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Let \( z = 0^i1^i \).

Then \( xz = 0^{2i+1} \in L \).

And \( yz = 0^{i+j}1^i \notin L \), because \( i + j \neq 2i \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

Solution:

For all non-negative integers \( i \neq j \), the strings \( 0^{2i} \) and \( 0^{2j} \) are distinguished by the suffix \( 1^i \), because \( 0^{2i+1} \in L \) but \( 0^{2j+1} \notin L \). Thus, the language \((00)^*\) is an infinite fooling set for \( L \).
Solution:
If $L$ were regular, then the language

$$(0 + 1)^* \setminus L \cap 0^*1^* = \{0^n1^n \mid m \neq 2n\}$$

would also be regular, because regular languages are closed under complement and intersection. But we just proved that $\{0^n1^n \mid m \neq 2n\}$ is not regular in problem 3. [Yes, this proof would be worth full credit, either in homework or on an exam.]

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5 Strings of properly nested parentheses (), brackets [], and braces {}. For example, the string ([]){} is in this language, but the string ([]) is not, because the left and right delimiters don’t match.

Solution:
Let $F$ be the language $\ast$.
Let $x$ and $y$ be arbitrary strings in $F$.
Then $x = i^i$ and $y = j^j$ for some non-negative integers $i \neq j$.
Let $z = \#0^i$.
Then $xz = (i^i)^i \in L$.
And $yz = (j^j)^i \notin L$, because $i \neq j$.
Thus, $F$ is a fooling set for $L$.
Because $F$ is infinite, $L$ cannot be regular.

Solution:
For any non-negative integers $i \neq j$, the strings $i^i$ and $j^j$ are distinguished by the suffix $i^i$, because $(i^i)^i \in L$ but $(j^j)^j \notin L$. Thus, the language $\ast$ is an infinite fooling set.

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6 Strings of the form $w_1\#w_2\# \cdots \#w_n$ for some $n \geq 2$, where each substring $w_i$ is a string in $\{0, 1\}^*$, and some pair of substrings $w_i$ and $w_j$ are equal.

Solution:
Let $F$ be the language $0^*$. 
Let $x$ and $y$ be arbitrary strings in $F$.
Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.
Let $z = \#0^i$.
Then $xz = 0^i\#0^i \in L$.
And $yz = 0^i\#0^i \notin L$, because $i \neq j$.
Thus, $F$ is a fooling set for $L$.
Because $F$ is infinite, $L$ cannot be regular.
For any non-negative integers \( i \neq j \), the strings \( 0^i \) and \( 0^j \) are distinguished by the suffix \( \#0^i \), because \( 0^i \#0^i \in L \) but \( 0^j \#0^i \notin L \). Thus, the language \( 0^* \) is an infinite fooling set.

Extra problems

7 \( \{0^{n^2} \mid n \geq 0\} \)

Solution:
Let \( x \) and \( y \) be distinct arbitrary strings in \( L \).
Without loss of generality, \( x = 0^{i^2} \) and \( y = 0^{j^2} \) for some \( i > j \geq 0 \).
Let \( z = 0^{2i+1} \).
Then \( xz = 0^{i^2 + 2i+1} = 0^{(i+1)^2} \in L \).
On the other hand, \( yz = 0^{j^2 + 2j+1} \notin L \), because \( i^2 < j^2 + 2j + 1 < (i+1)^2 \).
Thus, \( z \) distinguishes \( x \) and \( y \).
We conclude that \( L \) is an infinite fooling set for \( L \), so \( L \) cannot be regular.

Solution:
Let \( x \) and \( y \) be distinct arbitrary strings in \( 0^* \).
Without loss of generality, \( x = 0^i \) and \( y = 0^j \) for some \( i > j \geq 0 \).
Let \( z = 0^{i^2+i+1} \).
Then \( xz = 0^{i^2+2i+1} = 0^{(i+1)^2} \in L \).
On the other hand, \( yz = 0^{i^2+j+i+1} \notin L \), because \( i^2 < i^2 + i + j + 1 < (i+1)^2 \).
Thus, \( z \) distinguishes \( x \) and \( y \).
We conclude that \( 0^* \) is an infinite fooling set for \( L \), so \( L \) cannot be regular.

Solution:
Let \( x \) and \( y \) be distinct arbitrary strings in \( 0000^* \).
Without loss of generality, \( x = 0^i \) and \( y = 0^j \) for some \( i > j \geq 3 \).
Let \( z = 0^{i^2-i} \).
Then \( xz = 0^{i^2} \in L \).
On the other hand, \( yz = 0^{i^2-i+j} \notin L \), because
\[
(i - 1)^2 = i^2 - 2i + 1 < i^2 - i < i^2 - i + j < i^2.
\]
(The first inequalities requires \( i \geq 2 \), and the second \( j \geq 1 \).)
Thus, \( z \) distinguishes \( x \) and \( y \).
We conclude that \( 0000^* \) is an infinite fooling set for \( L \), so \( L \) cannot be regular.
\{ w \in (0 + 1)^* \mid w \text{ is the binary representation of a perfect square} \}

**Solution:**

We design our fooling set around numbers of the form \((2^k + 1)^2 = 2^{2k} + 2^{k+1} + 1 = 10^{k-2}10^k1 \in L\), for any integer \(k \geq 2\). The argument is somewhat simpler if we further restrict \(k\) to be even.

Let \(F = 1(00)^*1\), and let \(x\) and \(y\) be arbitrary strings in \(F\).

Then \(x = 10^{2i-2}1\) and \(y = 10^{2j-2}1\), for some positive integers \(i \neq j\).

Without loss of generality, assume \(i < j\). (Otherwise, swap \(x\) and \(y\).)

Let \(z = 0^{2i}1\).

Then \(xz = 10^{2i-2}10^{2i}1\) is the binary representation of \(2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2\), and therefore \(xz \in L\).

On the other hand, \(yz = 10^{2j-2}10^{2j}1\) is the binary representation of \(2^{2i+2j} + 2^{2i+1} + 1\). Simple algebra gives us the inequalities

\[
(2^{i+j})^2 = 2^{2i+2j} < 2^{2i+2j} + 2^{2i+1} + 1 < 2^{2(i+j) + i+j+1} + 1 = (2^{i+j} + 1)^2.
\]

So \(2^{2i+2j} + 2^{2i+1} + 1\) lies between two consecutive perfect squares, and thus is not a perfect square, which implies that \(yz \notin L\).

We conclude that \(F\) is a fooling set for \(L\). Because \(F\) is infinite, \(L\) cannot be regular.