

Prove that each of the following problems is NP-hard.

- 1 Given an undirected graph  $G$ , does  $G$  contain a simple path that visits all but 374 vertices?
- 2 Given an undirected graph  $G$ , does  $G$  have a spanning tree in which every node has degree at most 374?
- 3 Given an undirected graph  $G$ , does  $G$  have a spanning tree with at most 374 leaves?
- 4 Recall that a 5-coloring of a graph  $G$  is a function that assigns each vertex of  $G$  a “color” from the set  $\{0, 1, 2, 3, 4\}$ , such that for any edge  $uv$ , vertices  $u$  and  $v$  are assigned different “colors”. A 5-coloring is *careful* if the colors assigned to adjacent vertices are not only distinct, but differ by more than 1 (mod 5). Prove that deciding whether a given graph has a careful 5-coloring is NP-hard. (**Hint:** Reduce from the standard 5COLOR problem.)

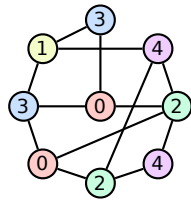


Figure 1: A careful 5-coloring.

- 5 Prove that the following problem is NP-hard: Given an undirected graph  $G$ , find *any* integer  $k > 374$  such that  $G$  has a proper coloring with  $k$  colors but  $G$  does not have a proper coloring with  $k - 374$  colors.
- 6 **To think about later:** A *bicoloring* of an undirected graph assigns each vertex a set of *two* colors. There are two types of bicoloring: In a *weak* bicoloring, the endpoints of each edge must use *different* sets of colors; however, these two sets may share one color. In a *strong* bicoloring, the endpoints of each edge must use *distinct* sets of colors; that is, they must use four colors altogether. Every strong bicoloring is also a weak bicoloring.
  - 6.A. Prove that finding the minimum number of colors in a weak bicoloring of a given graph is NP-hard.
  - 6.B. Prove that finding the minimum number of colors in a strong bicoloring of a given graph is NP-hard.

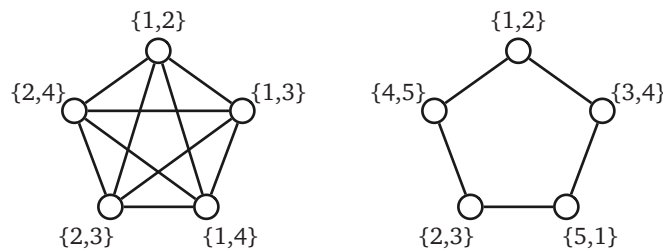


Figure 2: Left: A weak bicoloring of a 5-clique with four colors. Right A strong bicoloring of a 5-cycle with five colors.