Suppose you are given a magic black box that somehow answers the following decision problem in polynomial time:

- **Input**: A CNF formula \( \varphi \) with \( n \) variables \( x_1, x_2, \ldots, x_n \).
- **Output**: True if there is an assignment of True or False to each variable that satisfies \( \varphi \).

Using this black box as a subroutine, describe an algorithm that solves the following related search problem in polynomial time:

- **Input**: A CNF formula \( \varphi \) with \( n \) variables \( x_1, \ldots, x_n \).
- **Output**: A truth assignment to the variables that satisfies \( \varphi \), or \textit{None} if there is no satisfying assignment.

*(Hint: You can use the magic box more than once.)*

2. An \textit{independent set} in a graph \( G \) is a subset \( S \) of the vertices of \( G \), such that no two vertices in \( S \) are connected by an edge in \( G \). Suppose you are given a magic black box that somehow answers the following decision problem in polynomial time:

- **Input**: An undirected graph \( G \) and an integer \( k \).
- **Output**: True if \( G \) has an independent set of size \( k \), and False otherwise.

2.A. Using this black box as a subroutine, describe algorithms that solves the following optimization problem in polynomial time:

- **Input**: An undirected graph \( G \).
- **Output**: The size of the largest independent set in \( G \).

*(Hint: You have seen this problem before.)*

2.B. Using this black box as a subroutine, describe algorithms that solves the following search problem in polynomial time:

- **Input**: An undirected graph \( G \).
- **Output**: An independent set in \( G \) of maximum size.

To think about later:

3. Formally, a \textit{proper coloring} of a graph \( G = (V, E) \) is a function \( c: V \to \{1, 2, \ldots, k\} \), for some integer \( k \), such that \( c(u) \neq c(v) \) for all \( uv \in E \). Less formally, a valid coloring assigns each vertex of \( G \) a color, such that every edge in \( G \) has endpoints with different colors. The \textit{chromatic number} of a graph is the minimum number of colors in a proper coloring of \( G \).

Suppose you are given a magic black box that somehow answers the following decision problem in polynomial time:

- **Input**: An undirected graph \( G \) and an integer \( k \).
- **Output**: True if \( G \) has a proper coloring with \( k \) colors, and False otherwise.

Using this black box as a subroutine, describe an algorithm that solves the following \textit{coloring problem} in polynomial time:

- **Input**: An undirected graph \( G \).
- **Output**: A valid coloring of \( G \) using the minimum possible number of colors.

*(Hint: You can use the magic box more than once. The input to the magic box is a graph and only a graph, meaning \textbf{only} vertices and edges.)*