28 (100 pts.) Network deployment.

Consider an undirected connected graph $G$ that represents a map. Every vertex is a location, and every edge represents a road that connects the two locations, with the positive weight of the edge being the distance between the two locations. Let $d(x,y)$ denote the shortest path distance between any two vertices $x,y \in V(G)$. We have a set $X \subseteq V(G)$ of locations we would like to connect together via a common connected network. Computing the optimal such network is surprisingly difficult, but one can build a reasonably good network.

A natural strategy for deploying/building such a network is as follows – order the vertices of $X$ in some order $v_1,v_2,\ldots,v_t$, where $t = |X|$. In the $i$th step of the deployment, we install a server at $v_i$. Let $u_i$ be the closest server among $v_1,\ldots,v_{i-1}$ to $v_i$. We connect $v_i$ to $u_i$ (paying $d(v_i, u_i)$), and continue to the next location. The cost of the deployment is $\sum_{i=2}^{n} d(v_i, u_i)$.

28.A. (20 pts.) Consider the graph $G = (\{1, 2, \ldots, n\}, \{12, 23, \ldots, (n-1)n\})$, where the weight of all edges is 1. Show, that there is an ordering over the vertices of $G$ such that the deployment cost of $X = V(G)$ is $\Omega(n \log n)$.

28.B. (20 pts.) Let $T$ be the cheapest tree, such that $X \subseteq V(T)$. Verify that the cost of the tree $w(T) = \sum_{e \in T} w(e)$ is a lower bound on the deployment cost of $X$ for any ordering. Prove that there exists a closed walk that visits all the vertices of $X$, and the total weight of the edges of the walk is at most $2w(T)$.

28.C. (20 pts.) Let $x, y$ be the closest pair of vertices in $X$. Formally, it is one of the pairs realizing $\min_{u \in X} \min_{v \in X \setminus \{u\}} d(u, v)$. Prove, using the above, that $d(x, y) \leq 2w(T)/|X|$.

28.D. (40 pts.) Consider the greedy algorithm that computes the closest pair of vertices $x, y \in X$, sets $v_t = x$, and then recursively computes the ordering for $X \setminus \{x\}$. Let $\Pi$ be the resulting ordering of $X$. Prove that the deployment cost of $X$ is bounded by $O(w(T) \log n)$.

(Clearly, this algorithm can be implemented in polynomial time – but the exact running time here is unimportant.)

29 (100 pts.) Few spies.

Let $G = (C \cup S, E)$ be a graph with $n$ vertices and $m$ edges. The vertices of $C \cup S$ represents citizens in the glorious democratic republic of north Narnia (GDRNN). An edge between two people indicates that they are friends. The vertices of $S$ represents people that are willing to spy on their friends to see if they do any illegal activities (like enjoying themselves, etc). The government of GDRNN would like to choose a set of spies $S' \subseteq S$, of minimum size, such that every citizen in $C$ is connected to some vertex of $S'$ (the members of $S$ are trusted by the government – so no need to spy on them). Computing the smallest such set is surprisingly difficult. Again, we are going to be happy with a simple greedy strategy\(^1\).

\(^1\)Seeing the movie “The lives of others” might not help you solve this problem, but you might enjoy it anyway.
Let $C_0 = C$ and $S_0 = \emptyset$. In the $i$th iteration, for $i > 0$, we pick the vertex $s_i$ in $S$ that is connected to the largest number of citizens in $C_{i-1}$ (resolving ties arbitrarily). We then set $S_i = S_{i-1} \cup \{s_i\}$, and $C_i = C_{i-1} \setminus \Gamma(s_i)$, where $\Gamma(s_i)$ is the set of citizens connected to $s_i$. We stop as soon as $C_i$ is empty, and output $S_i$ as the desired set of spies.

29.A. (20 pts.) Assume that the optimal solution is of size $k$. Prove, that for any $i$, $s_i$ is connected to at least $|C_{i-1}| / k$ citizens in $C_{i-1}$ (hint: think about the $k$ optimal spies $o_1, \ldots, o_k$).

29.B. (20 pts.) Prove that $|C_i| \leq (1 - 1/k) |C_{i-1}|$.

29.C. (20 pts.) Using that $(1 - 1/k)^k \leq 1/e$, prove that for all $i$, we have that $|C_{i+k}| \leq |C_i| / e$.

29.D. (40 pts.) Prove, that the above algorithm outputs a set of at most $k(\lceil \ln n \rceil + 1)$ spies, such that all the citizens of $C$ are spied on by these spies.

30. (100 pts.) Undecidable, that’s what you are.

For each of the following languages, either prove that it is undecidable (by providing a detailed reduction from a known undecidable language), or describe an algorithm that decides this language – your description of the algorithm should be detailed and self contained. (Note, that you cannot use Rice Theorem in solving this problem.)

30.A. (25 pts.) $L_1 = \{ \langle M, N \rangle \mid L(M) = \overline{L(N)} \}$, where $M$ is a Turing machine, and $N$ is a NFA.

30.B. (25 pts.) $L_2 = \{ \langle N \rangle \mid L(N) \text{ is infinite} \}$, where $N$ is an NFA.

30.C. (25 pts.) $L_3 = \{ \langle R, N \rangle \mid L(R) = L(N) \}$, where $R$ is a regular expression, and $N$ is an NFA.

30.D. (25 pts.)

$L_4 = \{ \langle M \rangle \mid L(M) \text{ contains some word of even length} \}$, where $M$ is a Turing machine.