7 (100 pts.) Draw me a sheep.

For each of the following languages in 7.A–7.C., draw an NFA that accepts them. Your automata should have a small number of states. Provide a short explanation of your solution, if needed.

7.A. All strings in \{0, 1, 2\}∗ such that at least one of the symbols 0, 1, or 2 occurs at most 4 times. (Example: 120021220210 is in the language, since 1 occurs 4 times.)

7.B. \(((01)^* (10)^* + 00)^* (1 + 00 + \varepsilon) \cdot (11)^*\).

7.C. All strings in \{0, 1\}∗ such that the last symbol is the same as the third last symbol. (Example: 1100101 is in the language, since the last and the third last symbol are 1.)

7.D. Use the power-set construction (also called subset construction) to convert your NFA from 7.C. to a DFA. You may omit unreachable states.

8 (100 pts.) Fun with parity.

Given \(L \subseteq \{0, 1\}^\ast\), define \(even_0(L)\) to be the set of all strings in \{0, 1\}∗ that can be obtained by taking a string in \(L\) and inserting an even number of 0’s (anywhere in the string). Similarly, define \(odd_0(L)\) to be the set of all strings \(x\) in \{0, 1\}∗ that can be obtained by taking a string in \(L\) and inserting an odd number of 0’s.

(Example: if 01101 \(\in L\), then 0101000100 \(\in even_0(L)\).)

(Another example: if \(L\) is \(1^\ast\), then \(even_0(L)\) can be described by the regular expression \((1^*01^*0)^*1^\ast\).)

You will prove that if \(L \subseteq \{0, 1\}^\ast\) is regular, then \(even_0(L)\) and \(odd_0(L)\) are regular. Specifically, given a regular expression \(r\), you will describe a recursive algorithm to construct regular expressions for \(even_0(L(r))\) and \(odd_0(L(r))\).

8.A. For each of the base cases of regular expressions \(\emptyset, \varepsilon, 0,\) and \(1\), give regular expressions for \(even_0(L(r))\) and \(odd_0(L(r))\).

8.B. Given regular expressions for \(r_j' = even_0(L(r_j))\) and \(r_j'' = odd_0(L(r_j))\) for \(j \in \{1, 2\}\), give regular expressions for

(i) \(even_0(L(r_1 + r_2))\)
(ii) \(odd_0(L(r_1 + r_2))\)
(iii) \(even_0(L(r_1r_2))\)
(iv) \(odd_0(L(r_1r_2))\)
(v) \(even_0(L(r_1^\ast))\)
(vi) \(odd_0(L(r_1^\ast))\)

Give brief justification of correctness for each of the above.
Let binary(i) denote the binary representation of a positive integer i. (Note that the string binary(i) must start with a 1.)

Given a language $L \subseteq \{0, 1\}^*$, define $\text{increment}(L) = \{\text{binary}(i + 1) \mid \text{binary}(i) \in L\}$.

(Example: for $L = \{100, 101011, 111\}$, we have $\text{increment}(L) = \{101, 101100, 1000\}$.)

Prove that if $L$ is regular, then $\text{increment}(L)$ is regular. Specifically, given a DFA $M = (Q, \Sigma, \delta, s, A)$ for $L$, describe how to construct an NFA $M'$ for $\text{increment}(L)$. Prove correctness of your construction.