Which exam room to go to based on your discussion section.

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<tr>
<th>ECEB 1002</th>
<th>SC 1404</th>
<th>DCL 1320</th>
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<tr>
<td>AYA 9am Yipu</td>
<td>AYF 2pm Konstantinos</td>
<td>AYH 4pm Robert</td>
<td>BYB 10am Zhongyi</td>
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<td>AYB 10am Xilin</td>
<td>AYG 3pm Robert</td>
<td>AYK 2pm Shant</td>
<td>BYA 9am Zhongyi</td>
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<td>AYC 11am Xilin</td>
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<td>AYD noon Mitch</td>
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<td>AYE 1pm Ravi</td>
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Name: __________________________
NetID: __________________________

- Don’t panic!
- Please print your name, print your NetID, and circle your discussion section in the boxes above.
- There are five questions – you should answer all of them.
- If you brought anything except your writing implements, your double-sided handwritten (in the original) 8½" × 11" cheat sheet, and your university ID, please put it away for the duration of the exam. In particular, please turn off and put away all medically unnecessary electronic devices.
  - Submit your cheat sheet together with your exam. We will not return or scan the cheat sheets, so photocopy them before the exam if you want a copy.
  - If you are NOT using a cheat sheet, please indicate so in large friendly letters on this page.
- Please read all the questions before starting to answer them. Please ask for clarification if any question is unclear.
- This exam lasts 120 minutes. The clock started when you got the exam.
- If you run out of space for an answer, feel free to use the blank pages at the back of this booklet, but please tell us where to look.
- As usual, answering any (sub)problem with “I don’t know” (and nothing else) is worth 25% partial credit. Correct, complete, but slightly sub-optimal solutions are always worth more than 25%. Solutions that are exponentially (or significantly) slower than the expected solution would get no points at all. A blank answer is not the same as “I don’t know”.
- Total IDK points for the whole exam would not exceed 10.
- Give complete solutions, not examples. Declare all your variables. If you don’t know the answer admit it and use IDK. Write short concise answers.
- Style counts. Please use the backs of the pages or the blank pages at the end for scratch work, so that your actual answers are clear.
- Please return all paper with your answer booklet: your question sheet, your cheat sheet, and all scratch paper.
- Good luck!
1. (20 pts.) Short questions.

1.A. (10 pts.) Give an asymptotically tight solution to the following recurrence, where $T(n) = O(1)$ for $n < 10$, and otherwise:

$T(n) = T(2n/3) + T(n/2) + O(n^2)$.

1.B. (10 pts.) Given a directed graph $G$, describe a linear time algorithm that decides if there are three distinct vertices $x, y, z$, such that (i) there is a path from $x$ to $y$ in $G$, (ii) there is a path from $y$ to $z$ in $G$, and (iii) there is a path from $z$ to $x$ in $G$.

2. (20 pts.) Given a directed graph $G = (V, E)$ with positive edge lengths. Let $\ell(u, v)$ be the length of edge $(u, v) \in E$, and let $d(u, v)$ be the length of the shortest path from $u$ to $v$ in $G$. Given two nodes $s$ and $t$, there might be many different paths that realize the shortest path between $s$ and $t$, and let $\Pi$ be the set of all such paths. A vertex is useful if it lies on any path of $\Pi$. Describe how to compute, as fast as possible, all the useful vertices in $G$ (given $s$ and $t$). What is the running time of your algorithm.

3. (20 pts.) Suppose you are given a sorted array of $n$ distinct numbers that has been rotated right by $k$ steps, for some unknown integer $k$ between 1 and $n - 1$. That is, you are given an array $A[1...n]$ such that some prefix $A[1...k]$ is sorted in increasing order, the corresponding suffix $A[k+1...n]$ is also sorted in increasing order, and $A[n] < A[1]$. For example, the below array with $n=10$ has been rotated by $k=7$.

$$
egin{array}{cccccccccc}
35 & 65 & 108 & 197 & 303 & 499 & 833 & 3 & 4 & 19
\end{array}
$$

Given a number $x$, describe an algorithm, as fast as possible, that decides if $x$ appears somewhere in the $A$. What is the running time of your algorithm? Argue that your algorithm is correct.

4. (20 pts.) We are given a sequence of $n$ numbers $A[1], \ldots, A[n]$, and integers $g$ and $\ell$ with $\ell \geq n/g$. We want to choose a subsequence $A[i_1], \ldots, A[i_\ell]$ of length $\ell$, such that $i_1 = 1$, $i_\ell = n$, and $1 \leq i_{j+1} - i_j \leq g$ for all $j = 1, \ldots, \ell - 1$, while minimizing the sum $A[i_1] + \cdots + A[i_\ell]$.

Example: for the input sequence $0, 4, 3, 1, 11, 8, 5, 2$ and $g = 3$ and $\ell = 5$, we could pick $0 + 1 + 8 + 5 + 2 = 15$, but the optimal solution has sum $0 + 3 + 1 + 5 + 2 = 11$.

Describe an algorithm, as fast as possible, to compute the optimal sum, by using dynamic programming. (You do not need to output the optimal subsequence.) Give a clear English description of the function you are trying to evaluate, and how to call your function to get the final answer, then provide a recursive formula for evaluating the function (including base cases). If a correct evaluation order is specified clearly, pseudocode is not required. Analyze the running time as a function of $n$, $g$, and $\ell$. 


You are given a directed graph $G$ with $n$ vertices and $m$ edges ($m \geq n$), where each edge $e$ has an integer weight $w(e)$ (which could be positive or negative) and each vertex is marked “red” or “blue”. You are also given a (small) positive integer $b$.

Describe an algorithm, as fast as possible, to find a walk with the smallest total weight, such that the start vertex is red, the end vertex is red, and the number of blue vertices is divisible by $b$ (with no restrictions on the number of red vertices). Your solution should involve constructing a new graph and applying a known algorithm on this graph. Analyze the running time as a function of $n$, $m$, and $b$. 