

**“CS 374”: Algorithms and Models of Computation, Fall 2014
Final Exam — Version A — December 16, 2014**

Name:			
NetID:			
Section:	1	2	3

#	1	2	3	4	5	6	Total
Score							
Max	20	10	10	10	10	10	70
Grader							

- **Don't panic!**
- Please print your name and your NetID and circle your discussion section in the boxes above.
- This is a closed-book, closed-notes, closed-electronics exam. If you brought anything except your writing implements and your two double-sided 8½" × 11" cheat sheets, please put it away for the duration of the exam. In particular, you may not use *any* electronic devices.
- **Please read the entire exam before writing anything.** Please ask for clarification if any question is unclear.
- **You have 180 minutes.**
- If you run out of space for an answer, continue on the back of the page, or on the blank pages at the end of this booklet, **but please tell us where to look.** Alternatively, feel free to tear out the blank pages and use them as scratch paper.
- **Please return your cheat sheets and all scratch paper with your answer booklet.**
- If you use a greedy algorithm, you must prove that it is correct to receive credit. **Otherwise, proofs are required only if we specifically ask for them.**
- As usual, answering any (sub)problem with “I don't know” (and nothing else) is worth 25% partial credit. **Yes, even for problem 1.** Correct, complete, but suboptimal solutions are *always* worth more than 25%. A blank answer is not the same as “I don't know”.
- **Good luck!** And have a great winter break!

1. For each of the following questions, indicate *every* correct answer by marking the “Yes” box, and indicate *every* incorrect answer by marking the “No” box. **Assume $P \neq NP$.** If there is any other ambiguity or uncertainty, mark the “No” box. For example:

Yes	No	$2 + 2 = 4$
Yes	No	$x + y = 5$
Yes	No	3SAT can be solved in polynomial time.
Yes	No	Jeff is not the Queen of England.

There are 40 yes/no choices altogether, each worth $\frac{1}{2}$ point.

- (a) Which of the following statements is true for *every* language $L \subseteq \{0, 1\}^*$?

Yes	No	L is non-empty.
Yes	No	L is decidable or L is infinite (or both).
Yes	No	L is accepted by some DFA with 42 states if and only if L is accepted by some NFA with 42 states.
Yes	No	If L is regular, then $L \in NP$.
Yes	No	L is decidable if and only if its complement \bar{L} is undecidable.

- (b) Which of the following computational models can be simulated by a deterministic Turing machine with three read/write heads, with at most polynomial slow-down in time, assuming $P \neq NP$?

Yes	No	A Java program
Yes	No	A deterministic Turing machine with one head
Yes	No	A deterministic Turing machine with 3 tapes, each with 5 heads
Yes	No	A nondeterministic Turing machine with one head
Yes	No	A nondeterministic finite-state automaton (NFA)

(c) Which of the following languages are decidable?

Yes	No	\emptyset
Yes	No	$\{0^n 1^n 0^n 1^n \mid n \geq 0\}$
Yes	No	$\{\langle M \rangle \mid M \text{ is a Turing machine}\}$
Yes	No	$\{\langle M \rangle \mid M \text{ accepts } \langle M \rangle \cdot \langle M \rangle\}$
Yes	No	$\{\langle M \rangle \mid M \text{ accepts a finite number of non-palindromes}\}$
Yes	No	$\{\langle M \rangle \mid M \text{ accepts } \emptyset\}$
Yes	No	$\{\langle M, w \rangle \mid M \text{ accepts } w^R\}$
Yes	No	$\{\langle M, w \rangle \mid M \text{ accepts } w \text{ after at most } w ^2 \text{ transitions}\}$
Yes	No	$\{\langle M, w \rangle \mid M \text{ changes a blank on the tape to a non-blank, given input } w\}$
Yes	No	$\{\langle M, w \rangle \mid M \text{ changes a non-blank on the tape to a blank, given input } w\}$

(d) Let M be a standard Turing machine (with a single one-track tape and a single head) that **decides** the regular language 0^*1^* . Which of the following **must** be true?

Yes	No	Given an empty initial tape, M eventually halts.
Yes	No	M accepts the string 1111 .
Yes	No	M rejects the string 0110 .
Yes	No	M moves its head to the right at least once, given input 1100 .
Yes	No	M moves its head to the right at least once, given input 0101 .
Yes	No	M never accepts before reading a blank.
Yes	No	For some input string, M moves its head to the left at least once.
Yes	No	For some input string, M changes at least one symbol on the tape.
Yes	No	M always halts.
Yes	No	If M accepts a string w , it does so after at most $O(w ^2)$ steps.

(e) Consider the following pair of languages:

- $\text{HAMILTONIANPATH} := \{G \mid G \text{ contains a Hamiltonian path}\}$
- $\text{CONNECTED} := \{G \mid G \text{ is connected}\}$

Which of the following *must* be true, assuming $P \neq NP$?

- | | | |
|-----|----|---|
| Yes | No | $\text{CONNECTED} \in \text{NP}$ |
| Yes | No | $\text{HAMILTONIANPATH} \in \text{NP}$ |
| Yes | No | HAMILTONIANPATH is decidable. |
| Yes | No | There is no polynomial-time reduction from HAMILTONIANPATH to CONNECTED . |
| Yes | No | There is no polynomial-time reduction from CONNECTED to HAMILTONIANPATH . |

(f) Suppose we want to prove that the following language is undecidable.

$$\text{ALWAYSHALTS} := \{\langle M \rangle \mid M \text{ halts on every input string}\}$$

Rocket J. Squirrel suggests a reduction from the standard halting language

$$\text{HALT} := \{\langle M, w \rangle \mid M \text{ halts on inputs } w\}.$$

Specifically, suppose there is a Turing machine AH that decides ALWAYSHALTS . Rocky claims that the following Turing machine H decides HALT . Given an arbitrary encoding $\langle M, w \rangle$ as input, machine H writes the encoding $\langle M' \rangle$ of a new Turing machine M' to the tape and passes it to AH , where M' implements the following algorithm:

$M'(x)$:
 if M accepts w
 reject
 if M rejects w
 accept

Which of the following statements is true for all inputs $\langle M, w \rangle$?

- | | | |
|-----|----|---|
| Yes | No | If M accepts w , then M' halts on every input string. |
| Yes | No | If M diverges on w , then M' halts on every input string. |
| Yes | No | If M accepts w , then AH accepts $\langle M' \rangle$. |
| Yes | No | If M rejects w , then H rejects $\langle M, w \rangle$. |
| Yes | No | H decides the language HALT . (That is, Rocky's reduction is correct.) |

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2. A **relaxed 3-coloring** of a graph G assigns each vertex of G one of three colors (for example, red, green, and blue), such that **at most one** edge in G has both endpoints the same color.
- (a) Give an example of a graph that has a relaxed 3-coloring, but does not have a proper 3-coloring (where every edge has endpoints of different colors).
 - (b) **Prove** that it is NP-hard to determine whether a given graph has a relaxed 3-coloring.

3. Give a complete, formal, self-contained description of a DFA that accepts all strings in $\{0, 1\}^*$ containing at least ten 0s and at most ten 1s. Specifically:
- What are the states of your DFA?
 - What is the start state of your DFA?
 - What are the accepting states of your DFA?
 - What is your DFA's transition function?

4. Suppose you are given three strings $A[1..n]$, $B[1..n]$, and $C[1..n]$. Describe and analyze an algorithm to find the maximum length of a common subsequence of all three strings. For example, given the input strings

$$A = \text{AxxBxxCDxEF}, \quad B = \text{yyABCDyEyFy}, \quad C = \text{zAzzBCDzEFz},$$

your algorithm should output the number 6, which is the length of the longest common subsequence **ABCDEF**.

5. For each of the following languages over the alphabet $\Sigma = \{0, 1\}$, either *prove* that the language is regular, or *prove* that the language is not regular.

(a) $\{www \mid w \in \Sigma^*\}$

(b) $\{wxw \mid w, x \in \Sigma^*\}$

6. A **number maze** is an $n \times n$ grid of positive integers. A token starts in the upper left corner; your goal is to move the token to the lower-right corner. On each turn, you are allowed to move the token up, down, left, or right; the distance you may move the token is determined by the number on its current square. For example, if the token is on a square labeled 3, then you may move the token three steps up, three steps down, three steps left, or three steps right. However, you are never allowed to move the token off the edge of the board.

Describe and analyze an efficient algorithm that either returns the minimum number of moves required to solve a given number maze, or correctly reports that the maze has no solution. For example, given the maze shown below, your algorithm would return the number 8.

3	5	7	4	6
5	3	1	5	3
2	8	3	1	4
4	5	7	2	3
3	1	3	2	★

3	5	7	4	6
5	3	1	5	3
2	8	3	1	4
4	5	7	2	3
3	1	3	2	★

A 5×5 number maze that can be solved in eight moves.

(scratch paper)

(scratch paper)

You may assume the following problems are NP-hard:

- CIRCUITSAT:** Given a boolean circuit, are there any input values that make the circuit output TRUE?
- 3SAT:** Given a boolean formula in conjunctive normal form, with exactly three literals per clause, does the formula have a satisfying assignment?
- MAXINDEPENDENTSET:** Given an undirected graph G , what is the size of the largest subset of vertices in G that have no edges among them?
- MAXCLIQUE:** Given an undirected graph G , what is the size of the largest complete subgraph of G ?
- MINVERTEXCOVER:** Given an undirected graph G , what is the size of the smallest subset of vertices that touch every edge in G ?
- 3COLOR:** Given an undirected graph G , can its vertices be colored with three colors, so that every edge touches vertices with two different colors?
- HAMILTONIANPATH:** Given an undirected graph G , is there a path in G that visits every vertex exactly once?
- HAMILTONIANCYCLE:** Given an undirected graph G , is there a cycle in G that visits every vertex exactly once?
- DIRECTEDHAMILTONIANCYCLE:** Given a directed graph G , is there a directed cycle in G that visits every vertex exactly once?
- TRAVELINGSALESMAN:** Given a graph G (either directed or undirected) with weighted edges, what is the minimum total weight of any Hamiltonian path/cycle in G ?
- DRAUGHTS:** Given an $n \times n$ international draughts configuration, what is the largest number of pieces that can (and therefore must) be captured in a single move?
- SUPER MARIO:** Given an $n \times n$ level for Super Mario Brothers, can Mario reach the castle?

You may assume the following languages are undecidable:

- $SELFREJECT := \{ \langle M \rangle \mid M \text{ rejects } \langle M \rangle \}$
- $SELFACCEPT := \{ \langle M \rangle \mid M \text{ accepts } \langle M \rangle \}$
- $SELFHALT := \{ \langle M \rangle \mid M \text{ halts on } \langle M \rangle \}$
- $SELFDIVERGE := \{ \langle M \rangle \mid M \text{ does not halt on } \langle M \rangle \}$
- $REJECT := \{ \langle M, w \rangle \mid M \text{ rejects } w \}$
- $ACCEPT := \{ \langle M, w \rangle \mid M \text{ accepts } w \}$
- $HALT := \{ \langle M, w \rangle \mid M \text{ halts on } w \}$
- $DIVERGE := \{ \langle M, w \rangle \mid M \text{ does not halt on } w \}$
- $NEVERREJECT := \{ \langle M \rangle \mid REJECT(M) = \emptyset \}$
- $NEVERACCEPT := \{ \langle M \rangle \mid ACCEPT(M) = \emptyset \}$
- $NEVERHALT := \{ \langle M \rangle \mid HALT(M) = \emptyset \}$
- $NEVERDIVERGE := \{ \langle M \rangle \mid DIVERGE(M) = \emptyset \}$