Proving that a problem \( X \) is NP-hard requires several steps:

- Choose a problem \( Y \) that you already know is NP-hard.
- Describe a reduction \( f \) from \( Y \) to \( X \), i.e., given input \( w \) for problem \( Y \), \( f(w) \) is an input to problem \( X \).
- Prove that the function \( f \) is computable in polynomial time, by outlining an algorithm running in polynomial time that computes \( f \).
- Prove that your reduction \( f \) is correct. This almost always requires two separate steps:
  - Prove that if \( w \in Y \) then \( f(w) \in X \), i.e., the reduction \( f \) transforms “yes” instances of \( Y \) into “yes” instances of \( X \).
  - Prove that if \( w \notin Y \) then \( f(w) \notin X \), i.e., the reduction \( f \) transforms “no” instances of \( Y \) into “no” instances of \( X \). Equivalently: Prove that if \( f(w) \in X \) then \( w \in Y \).

Proving that \( X \) is NP-Complete requires you to additionally prove that \( X \in \text{NP} \) by describing a non-deterministic polynomial-time algorithm for \( X \). Typically this is not hard for the problems we consider but it is not always obvious.

**Problem 1.** [Category: Proof] Recall the following \textsc{kColor} problem: Given an undirected graph \( G \), can its vertices be colored with \( k \) colors, so that the endpoints of every edge get different colors?

1. Describe a direct polynomial-time reduction from \textsc{3Color} to \textsc{4Color}. \textit{Hint}: Your reduction will take a graph \( G \) and output another graph \( G' \) such that \( G' \) is 4-colorable if and only if \( G \) is 3-colorable. You should think how an explicit 4-coloring for \( G' \) would enable you to obtain an explicit 3-coloring for \( G \).

2. Prove that \textsc{kColor} problem is NP-hard for any \( k \geq 3 \), by showing that \textsc{3Color} \( \leq \text{P} \) \textsc{kColor}, for \( k \geq 3 \).

**Problem 2.** [Category: Proof] Describe a polynomial-time reduction from \textsc{3Color} to \textsc{Sat}. Can you generalize it to reduce \textsc{kColor} to \textsc{Sat}. \textit{Hint}: Use a variable \( x(v, i) \) to indicate that \( v \) is colored \( i \) and express the constraints using clauses in CNF form.

**Problem 3.** [Category: Proof] Let \( G = (V, E) \) be a directed graph with edge lengths \( \ell(e), e \in E \). The lengths can be positive or negative. The Zero-Length-Cycle Problem is to decide whether \( G \) has a cycle \( C \) of length exactly equal to 0. Prove that this problem is NP-Complete. \textit{Hint: reduce Hamiltonian Path to Zero-Length-Cycle}