Give context-free grammars for each of the following languages.

1. $\{0^{2n}1^n \mid n \geq 0\}$

Solution: $S \rightarrow \varepsilon \mid 00S1$

2. $\{0^n1^n \mid m \neq 2n\}$  \[Hint: If m \neq 2n, then either m < 2n or m > 2n.\]

Solution: To simplify notation, let $\Delta(w) = \#(0, w) - 2\#(1, w)$. Our solution follows the following logic. Let $w$ be an arbitrary string in this language.

- Because $\Delta(w) \neq 0$, then either $\Delta(w) > 0$ or $\Delta(w) < 0$.
- If $\Delta(w) > 0$, then $w = 0^iz$ for some integer $i > 0$ and some suffix $z$ with $\Delta(z) = 0$.
- If $\Delta(w) < 0$, then $w = x1^j$ for some integer $j > 0$ and some prefix $x$ with either $\Delta(x) = 0$ or $\Delta(x) = 1$.
- Substrings with $\Delta = 0$ is generated by the previous grammar; we need only a small tweak to generate substrings with $\Delta = 1$.

We can encode this case analysis as a CFG as follows. The nonterminals $M$ and $L$ generate all strings where the number of $0$s is More or Less than twice the number of $1$s, respectively. The last nonterminal generates strings with $\Delta = 0$ or $\Delta = 1$.

$$S \rightarrow M \mid L \quad \{0^m1^n \mid m \neq 2n\}$$

$$M \rightarrow 0M \mid 0E \quad \{0^m1^n \mid m > 2n\}$$

$$L \rightarrow L1 \mid E1 \quad \{0^m1^n \mid m < 2n\}$$

$$E \rightarrow \varepsilon \mid 0 \mid 00E1 \quad \{0^m1^n \mid m = 2n \text{ or } 2n + 1\}$$

Solution: Here is another way to encode the logic of the previous solution as a CFG. We either identify a non-empty prefix of $0$s or a non-empty prefix of $1$s, so that the rest of the string as “balanced” as possible. We also generate strings with $\Delta = 1$ using a separate non-terminal.

$$S \rightarrow AE \mid EB \mid FB \quad \{0^m1^n \mid m \neq 2n\}$$

$$A \rightarrow 0 \mid 0A \quad 0^+ = \{0^i \mid i \geq 1\}$$

$$B \rightarrow 1 \mid 1B \quad 1^+ = \{1^j \mid j \geq 1\}$$

$$E \rightarrow \varepsilon \mid 00E1 \quad \{0^m1^n \mid m = 2n\}$$

$$F \rightarrow E1 \quad \{0^m1^n \mid m = 2n + 1\}$$
Solution: Here is yet another way to encode the logic of the first solution as a CFG. We separately generate all strings of the form \(0^{\text{odd}}1^n\), so that we don’t have to worry about the case \(\Delta = 1\) separately.

\[
S \rightarrow D | M | L \quad \{0^m 1^n \mid m \neq 2n\} \\
D \rightarrow 0 | 00D | D1 \quad \{0^m 1^n \mid m \text{ is odd}\} \\
M \rightarrow 0M | 0E \quad \{0^m 1^n \mid m > 2n\} \\
L \rightarrow L1 | E1 \quad \{0^m 1^n \mid m < 2n \text{ and } m \text{ is even}\} \\
E \rightarrow \varepsilon | 00E1 \quad \{0^m 1^n \mid m = 2n\}
\]

Solution: Intuitively, we can parse any string \(w \in L\) as follows. First, remove the first \(2k\) \(0\)s and the last \(k\) \(1\)s, for the largest possible value of \(k\). The remaining string cannot be empty, and it must consist entirely of \(0\)s, entirely of \(1\)s, or a single \(0\) followed by \(1\)s.

\[
S \rightarrow 00S1 | A | B | C \quad \{0^m 1^n \mid m \neq 2n\} \\
A \rightarrow 0 | 0A \quad 0^+ \\
B \rightarrow 1 | 1B \quad 1^+ \\
C \rightarrow 0 | 0B \quad 01^*
\]

3. \(\{0, 1\}^* \setminus \{0^{2n}1^n \mid n \geq 0\}\)

Solution: This language is the union of the previous language and the complement of \(0^*1^*\), which is \((0 + 1)^*10(0 + 1)^*\).

\[
S \rightarrow T | X \quad \{0, 1\}^* \setminus \{0^{2n}1^n \mid n \geq 0\} \\
T \rightarrow 00T1 | A | B | C \quad \{0^m 1^n \mid m \neq 2n\} \\
A \rightarrow 0 | 0A \quad 0^+ \\
B \rightarrow 1 | 1B \quad 1^+ \\
C \rightarrow 0 | 0B \quad 01^* \\
X \rightarrow Z10Z \quad (0 + 1)^*10(0 + 1)^* \\
Z \rightarrow \varepsilon | 0Z | 1Z \quad (0 + 1)^*
\]
Work on these later:

4. \( \{ w \in \{ 0, 1 \}^* \mid \#(0, w) = 2 \cdot \#(1, w) \} \) — Binary strings where the number of 0s is exactly twice the number of 1s.

**Solution:**

\[ S \rightarrow \varepsilon \mid SS \mid 00S1 \mid 1S00 \mid 0S1S0. \]

For any string \( w \), let \( \Delta(w) = \#(0, w) - 2 \cdot \#(1, w) \). Suppose \( w \) is a binary string such that \( \Delta(w) = 0 \). Suppose \( w \) is nonempty and has no non-empty proper prefix \( x \) such that \( \Delta(x) = 0 \). There are three possibilities to consider:

- Suppose \( \Delta(x) > 0 \) for every proper prefix \( x \) of \( w \). In this case, \( w \) must start with 00 and end with 1. Thus, \( w = 00x1 \) for some string \( x \in L \).
- Suppose \( \Delta(x) < 0 \) for every proper prefix \( x \) of \( w \). In this case, \( w \) must start with 1 and end with 00. Let \( x \) be the shortest non-empty prefix with \( \Delta(x) = 1 \). Thus, \( w = 1x00 \) for some string \( x \in L \).
- Finally, suppose \( \Delta(x) > 0 \) for some prefix \( x \) and \( \Delta(x') < 0 \) for some longer proper prefix \( x' \). Let \( x' \) be the shortest non-empty proper prefix of \( w \) with \( \Delta < 0 \). Then \( x' = 0y1 \) for some substring \( y \) with \( \Delta(y) = 0 \), and thus \( w = 0y1z0 \) for some strings \( y, z \in L \).

5. \( \{ 0, 1 \}^* \setminus \{ ww \mid w \in \{ 0, 1 \}^* \} \).

**Solution:**

All strings of odd length are in \( L \).

Let \( w \) be any even-length string in \( L \), and let \( m = \frac{|w|}{2} \). For some index \( i \leq m \), we have \( w_i \neq w_{m+i} \). Thus, \( w \) can be written as either \( x1y0z \) or \( x0y1z \) for some substrings \( x, y, z \) such that \( |x| = i - 1, |y| = m - 1, \) and \( |z| = m - i \). We can further decompose \( y \) into a prefix of length \( i - 1 \) and a suffix of length \( m - i \). So we can write any even-length string \( w \in L \) as either \( x1x'z'0z \) or \( x0x'z'1z \), for some strings \( x, x', z, z' \) with \( |x| = |x'| = i - 1 \) and \( |z| = |z'| = m - i \).

Said more simply, we can divide \( w \) into two odd-length strings, one with a 0 at its center, and the other with a 1 at its center.

\[
\begin{align*}
S \rightarrow AB & \mid BA \mid A \mid B & \text{strings not of the form } ww \\
A \rightarrow 0 & \mid \Sigma\Sigma & \text{odd-length strings with } 0 \text{ at center} \\
B \rightarrow 1 & \mid \Sigma\Sigma & \text{odd-length strings with } 1 \text{ at center} \\
\Sigma \rightarrow 0 & \mid 1 & \text{single character}
\end{align*}
\]