Prove that each of the following languages is not regular.

1. \(\{0^{2^n} \mid n \geq 0\}\)

**Solution (\(F = L\):)** Let \(F = L = \{0^{2^n} \mid n \geq 0\}\).

Let \(x\) and \(y\) be arbitrary distinct elements of \(F\).

Then \(x = 0^{2^i}\) and \(y = 0^{2^j}\) for some non-negative integers \(i \neq j\).

Let \(z = 0^{2^i}\).

Then \(xz = 0^{2^i}0^{2^i} = 0^{2^i+1} \in L\).

But \(yz = 0^{2^j}0^{2^j} = 0^{2^j+2^j} \notin L\), because \(i \neq j\).

Thus, \(F\) is a fooling set for \(L\).

Because \(F\) is infinite, \(L\) cannot be regular.

**Solution (\(F = 0^*\)):** Let \(F = 0^* = \{0^n \mid n \geq 0\}\).

Let \(x\) and \(y\) be arbitrary distinct elements of \(F\).

Then \(x = 0^i\) and \(y = 0^j\) for some non-negative integers \(i \neq j\).

Without loss of generality, assume \(i < j\).

Let \(r\) be any integer such that \(2^r > 2j\), and let \(z = 0^{2^r-j}\).

Then \(xz = 0^i0^{2^r-j} = 0^{2^r-j+i} \notin L\), because \(2^r > 2^r-j+i > 2^r-j > 2^r-2^{r-1} = 2^{r-1}\).

But \(yz = 0^j0^{2^r-j} = 0^{2^r} \in L\).

Thus, \(F\) is a fooling set for \(L\).

Because \(F\) is infinite, \(L\) cannot be regular.
2. $\{0^{2n}1^n \mid n \geq 0\}$

**Solution ($F = (00)^*$):** Let $F$ be the language $(00)^*$.

Let $x$ and $y$ be arbitrary distinct strings in $F$.

Then $x = 0^{2i}$ and $y = 0^{2j}$ for some non-negative integers $i \neq j$.

Let $z = 1^i$.

Then $xz = 0^{2i}1^i \in L$.

And $yz = 0^{2j}1^i \notin L$, because $i \neq j$.

Thus, $F$ is a fooling set for $L$.

Because $F$ is infinite, $L$ cannot be regular. ■

**Solution ($F = 0^*$):** Let $F$ be the language $0^*$.

Let $x$ and $y$ be arbitrary distinct strings in $F$.

Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.

Let $z = 0^i 1^i$.

Then $xz = 0^{2i}1^i \in L$.

And $yz = 0^{i+j}1^i \notin L$, because $i + j \neq 2i$.

Thus, $F$ is a fooling set for $L$.

Because $F$ is infinite, $L$ cannot be regular. ■
3. \{0^m1^n \mid m \neq 2n\}

**Solution (F = (\emptyset\emptyset)^*)**: Let F be the language (\emptyset\emptyset)^*.

Let x and y be arbitrary distinct strings in F.
Then x = 0^{2i} and y = 0^{2j} for some non-negative integers i \neq j.
Let z = 1^i.
Then \(xz = 0^{2i}1^i \not\in L\).
And \(yz = 0^{2j}1^i \in L\), because i \neq j.
Thus, F is a fooling set for L.
Because F is infinite, L cannot be regular.

**Solution (F = 0^*)**: Let F be the language 0^*.

Let x and y be arbitrary distinct strings in F.
Then x = 0^i and y = 0^j for some non-negative integers i \neq j.
Let z = 0^i1^i.
Then \(xz = 0^{2i}1^i \not\in L\).
And \(yz = 0^{i+j}1^i \in L\), because i + j \neq 2i.
Thus, F is a fooling set for L.
Because F is infinite, L cannot be regular.

**Solution (closure properties)**: If L were regular, then the language
\[(\emptyset + 1)^* \setminus \{0^m1^n \mid m = 2n\} = \{0^{2n}1^n \mid n \geq 0\}\]
would also be regular, because regular languages are closed under complement. But we just proved that \{0^{2n}1^n \mid n \geq 0\} is not regular in problem 2. [Yes, this proof would be worth full credit, either in homework or on an exam.]
4. Strings over \{0, 1\} where the number of 0s is exactly twice the number of 1s.

Solution \((F = 1^*)\): Let \(F\) be the language \(1^*\).

Let \(x\) and \(y\) be arbitrary distinct strings in \(F\).

Then \(x = 1^i\) and \(y = 1^j\) for some non-negative integers \(i \neq j\).

Let \(z = 0^{2i}\).

Then \(xz = 1^i0^{2i} \in L\).

And \(yz = 1^i0^{2j} \notin L\), because \(i \neq j\).

Thus, \(F\) is a fooling set for \(L\).

Because \(F\) is infinite, \(L\) cannot be regular. 

Solution \((F = 0^*)\): Let \(F\) be the language \(0^*\).

Let \(x\) and \(y\) be arbitrary distinct strings in \(F\).

Then \(x = 0^i\) and \(y = 0^j\) for some non-negative integers \(i \neq j\).

Let \(z = 0^i1^i\).

Then \(xz = 0^{2i}1^i \in L\).

And \(yz = 0^{i+j}1^i \notin L\), because \(i + j \neq 2i\).

Thus, \(F\) is a fooling set for \(L\).

Because \(F\) is infinite, \(L\) cannot be regular. 

Solution (closure properties): If \(L\) were regular, then the language

\[
L \cap 0^*1^* = \{ 0^{2n}1^n \mid n \geq 0 \}
\]

would also be regular, because regular languages are closed under intersection. But we just proved that \(\{ 0^{2n}1^n \mid n \geq 0 \}\) is not regular in problem 2. [Yes, this proof would be worth full credit, either in homework or on an exam.]
5. Strings of properly nested parentheses ( ), brackets [ ], and braces { }. For example, the string ( [ ] ) {} is in this language, but the string ( [ ] ) is not, because the left and right delimiters don’t match.

Solution: Let $F$ be the language $\{^*\}$.

Let $x$ and $y$ be arbitrary distinct strings in $F$.

Then $x = (^i)$ and $y = (^j)$ for some non-negative integers $i \neq j$.

Let $z = (^i)$.

Then $xz = (^i)i \in L$.

And $yz = (^j)i \notin L$, because $i \neq j$.

Thus, $F$ is a fooling set for $L$.

Because $F$ is infinite, $L$ cannot be regular.

Solution (closure properties): If $L$ were regular, then the language

\[ L' := L \cap \{^*\}^* = \{(^n)^n \mid n \geq 0\} \]

would also be regular, because regular languages are closed under intersection. But $L'$ is the same as the language $\{0^n1^n \mid n \geq 0\}$, except for renaming the symbols $0 \mapsto ( \text{ and } 1 \mapsto )$, and we proved that $\{0^n1^n \mid n \geq 0\}$ in class.

[Yes, this proof would be worth full credit, either in homework or on an exam.]

Work on these later:

6. Strings of the form $w_1 \# w_2 \# \cdots \# w_n$ for some $n \geq 2$, where each substring $w_i$ is a string in $\{0, 1\}^*$, and some pair of substrings $w_i$ and $w_j$ are equal.

Solution (make $n = 2$): Let $F$ be the language $\emptyset^*$.

Let $x$ and $y$ be arbitrary distinct strings in $F$.

Then $x = \emptyset^i$ and $y = \emptyset^j$ for some non-negative integers $i \neq j$.

Let $z = \# \emptyset^i$.

Then $xz = \emptyset^i \# \emptyset^i \in L$.

And $yz = \emptyset^j \# \emptyset^i \notin L$, because $i \neq j$.

Thus, $F$ is a fooling set for $L$.

Because $F$ is infinite, $L$ cannot be regular.
7. \( \{ \theta^{n^2} \mid n \geq 0 \} \)

**Solution:** Let \( x \) and \( y \) be arbitrary distinct strings in \( L \).

Without loss of generality, \( x = \theta^{i^2} \) and \( y = \theta^{j^2} \) for some \( i > j \geq 0 \).

Let \( z = \theta^{2i+1} \).

Then \( xz = \theta^{i^2+2i+1} = \theta^{(i+1)^2} \in L \).

On the other hand, \( yz = \theta^{i^2+2j+1} \notin L \), because \( i^2 < i^2 + 2j + 1 < (i+1)^2 \).

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( L \) is a fooling set for \( L \).

Because \( L \) is infinite, \( L \) cannot be regular.

**Solution:** Let \( x \) and \( y \) be arbitrary distinct strings in \( 0^* \).

Without loss of generality, \( x = \theta^i \) and \( y = \theta^j \) for some \( i > j \geq 0 \).

Let \( z = \theta^{i^2-i+1} \).

Then \( xz = \theta^{i^2+2i-1} = \theta^{(i+1)^2} \in L \).

On the other hand, \( yz = \theta^{i^2+i+j-1} \notin L \), because \( i^2 < i^2 + i + j - 1 < (i+1)^2 \).

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( 0^* \) is a fooling set for \( L \).

Because \( 0^* \) is infinite, \( L \) cannot be regular.

**Solution:** Let \( x \) and \( y \) be arbitrary distinct strings in \( 0000^* \).

Without loss of generality, \( x = \theta^i \) and \( y = \theta^j \) for some \( i > j \geq 3 \).

Let \( z = \theta^{i^2-i} \).

Then \( xz = \theta^{i^2} \in L \).

On the other hand, \( yz = \theta^{i^2-i+j} \notin L \), because

\[
(i - 1)^2 = i^2 - 2i + 1 < i^2 - i < i^2 - i + j < i^2.
\]

(The first inequality requires \( i \geq 2 \), and the second requires \( j \geq 1 \).)

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( 0000^* \) is a fooling set for \( L \).

Because \( 0000^* \) is infinite, \( L \) cannot be regular.
8. \{w \in (0 + 1)^* \mid w \text{ is the binary representation of a perfect square}\}

**Solution:** We design our fooling set around numbers of the form $(2^k + 1)^2 = 2^{2k} + 2^{k+1} + 1 = 10^{k-2}10^k1 \in L$, for any integer $k \geq 2$. The argument is somewhat simpler if we further restrict $k$ to be even.

Let $F = 1(00)^*1$, and let $x$ and $y$ be arbitrary distinct strings in $F$.

Then $x = 10^{2i-2}1$ and $y = 10^{2j-2}1$, for some positive integers $i \neq j$.

Without loss of generality, assume $i < j$. (Otherwise, swap $x$ and $y$.)

Let $z = 0^{2i}1$.

Then $xz = 10^{2i-2}10^{2i}1$ is the binary representation of $2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2$, and therefore $xz \in L$.

On the other hand, $yz = 10^{2j-2}10^{2j}1$ is the binary representation of $2^{2i+2j} + 2^{2i+1} + 1$. Simple algebra gives us the inequalities

\[
(2^{i+j})^2 = 2^{2i+2j} < 2^{2i+2j} + 2^{2i+1} + 1 < 2^{2(i+j)} + 2^{i+j+1} + 1 = (2^{i+j} + 1)^2.
\]

So $2^{2i+2j} + 2^{2i+1} + 1$ lies between two consecutive perfect squares, and thus is not a perfect square, which implies that $yz \notin L$.

We conclude that $F$ is a fooling set for $L$.

Because $F$ is infinite, $L$ cannot be regular.