1. Recall the following \textit{kColor} problem: Given an undirected graph \( G \), can its vertices be colored with \( k \) colors, so that every edge touches vertices with two different colors?

(a) Describe a direct polynomial-time reduction from \textsc{3Color} to \textsc{4Color}.

\begin{solution}
Suppose we are given an arbitrary graph \( G \). Let \( H \) be the graph obtained from \( G \) by adding a new vertex \( a \) (called an \textit{apex}) with edges to every vertex of \( G \). I claim that \( G \) is \textit{3}-colorable if and only if \( H \) is \textit{4}-colorable.

\( \implies \) Suppose \( G \) is \textit{3}-colorable. Fix an arbitrary \textit{3}-coloring of \( G \), and call the colors “red”, “green”, and “blue”. Assign the new apex \( a \) the color “plaid”. Let \( uv \) be an arbitrary edge in \( H \).

- If both \( u \) and \( v \) are vertices in \( G \), they have different colors.
- Otherwise, one endpoint of \( uv \) is plaid and the other is not, so \( u \) and \( v \) have different colors.

We conclude that we have a valid \textit{4}-coloring of \( H \), so \( H \) is \textit{4}-colorable.

\( \impliedby \) Suppose \( H \) is \textit{4}-colorable. Fix an arbitrary \textit{4}-coloring; call the apex’s color “plaid” and the other three colors “red”, “green”, and “blue”. Each edge \( uv \) in \( G \) is also an edge of \( H \) and therefore has endpoints of two different colors. Each vertex \( v \) in \( G \) is adjacent to the apex and therefore cannot be plaid.

We conclude that by deleting the apex, we obtain a valid \textit{3}-coloring of \( G \), so \( G \) is \textit{3}-colorable.

We can easily transform \( G \) into \( H \) in polynomial time by brute force.
\end{solution}
(b) Prove that $k$\textsc{Color} problem is NP-hard for any $k \geq 3$.

\textbf{Solution (direct)}: The lecture notes include a proof that 3\textsc{Color} is NP-hard. For any integer $k > 3$, I'll describe a direct polynomial-time reduction from 3\textsc{Color} to $k$\textsc{Color}.

Let $G$ be an arbitrary graph. Let $H$ be the graph obtained from $G$ by adding $k - 3$ new vertices $a_1, a_2, \ldots, a_{k-3}$, each with edges to every other vertex in $H$ (including the other $a_i$'s). I claim that $G$ is 3-colorable if and only if $H$ is $k$-colorable.

$\Rightarrow$ Suppose $G$ is 3-colorable. Fix an arbitrary 3-coloring of $G$. Color the new vertices $a_1, a_2, \ldots, a_{k-3}$ with $k - 3$ new distinct colors. Every edge in $H$ is either an edge in $G$ or uses at least one new vertex $a_i$; in either case, the endpoints of the edge have different colors. We conclude that $H$ is $k$-colorable.

$\Leftarrow$ Suppose $H$ is $k$-colorable. Each vertex $a_i$ is adjacent to every other vertex in $H$, and therefore is the only vertex of its color. Thus, the vertices of $G$ use only three distinct colors. Every edge of $G$ is also an edge of $H$, so its endpoints have different colors. We conclude that the induced coloring of $G$ is a proper 3-coloring, so $G$ is 3-colorable.

Given $G$, we can construct $H$ in polynomial time by brute force.

\textbf{Solution (induction)}: Let $k$ be an arbitrary integer with $k \geq 3$. Assume that $j$\textsc{Color} is NP-hard for every integer $3 \leq j < k$. There are two cases to consider.

- If $k = 3$, then $k$\textsc{Color} is NP-hard by the reduction from 3\textsc{Sat} in the lecture notes.

- Suppose $k > 3$. The reduction in part (a) directly generalizes to a polynomial-time reduction from $(k-1)$\textsc{Color} to $k$\textsc{Color}: To decide whether an arbitrary graph $G$ is $(k-1)$-colorable, add an apex and ask whether the resulting graph is $k$-colorable. The induction hypothesis implies that $(k-1)$\textsc{Color} is NP-hard, so the reduction implies that $k$\textsc{Color} is NP-hard.

In both cases, we conclude that $k$\textsc{Color} is NP-hard.
2. A Hamiltonian cycle in a graph $G$ is a cycle that goes through every vertex of $G$ exactly once. Deciding whether an arbitrary graph contains a Hamiltonian cycle is NP-hard.

A tonian cycle in a graph $G$ is a cycle that goes through at least half of the vertices of $G$. Prove that deciding whether a graph contains a tonian cycle is NP-hard.

**Solution (duplicate the graph):** I'll describe a polynomial-time reduction from HamiltonianCycle. Let $G$ be an arbitrary graph. Let $H$ be a graph consisting of two disjoint copies of $G$, with no edges between them; call these copies $G_1$ and $G_2$. I claim that $G$ has a Hamiltonian cycle if and only if $H$ has a tonian cycle.

$\implies$ Suppose $G$ has a Hamiltonian cycle $C$. Let $C_1$ be the corresponding cycle in $G_1$. $C_1$ contains exactly half of the vertices of $H$, and thus is a tonian cycle in $H$.

$\impliedby$ On the other hand, suppose $H$ has a tonian cycle $C$. Because there are no edges between the subgraphs $G_1$ and $G_2$, this cycle must lie entirely within one of these two subgraphs. $G_1$ and $G_2$ each contain exactly half the vertices of $H$, so $C$ must also contain exactly half the vertices of $H$, and thus is a Hamiltonian cycle in either $G_1$ or $G_2$. But $G_1$ and $G_2$ are just copies of $G$. We conclude that $G$ has a Hamiltonian cycle.

Given $G$, we can construct $H$ in polynomial time by brute force.

**Solution (add $n$ new vertices):** I'll describe a polynomial-time reduction from HamiltonianCycle. Let $G$ be an arbitrary graph, and suppose $G$ has $n$ vertices. Let $H$ be a graph obtained by adding $n$ new vertices to $G$, but no additional edges. I claim that $G$ has a Hamiltonian cycle if and only if $H$ has a tonian cycle.

$\implies$ Suppose $G$ has a Hamiltonian cycle $C$. Then $C$ visits exactly half the vertices of $H$, and thus is a tonian cycle in $H$.

$\impliedby$ On the other hand, suppose $H$ has a tonian cycle $C$. This cycle cannot visit any of the new vertices, so it must lie entirely within the subgraph $G$. Since $G$ contains exactly half the vertices of $H$, the cycle $C$ must visit every vertex of $G$, and thus is a Hamiltonian cycle in $G$.

Given $G$, we can construct $H$ in polynomial time by brute force.
To think about later:

3. Let $G$ be an undirected graph with weighted edges. A Hamiltonian cycle in $G$ is heavy if the total weight of edges in the cycle is at least half of the total weight of all edges in $G$. Prove that deciding whether a graph contains a heavy Hamiltonian cycle is NP-hard.

Solution (two new vertices): I’ll describe a polynomial-time a reduction from the Hamiltonian path problem. Let $G$ be an arbitrary undirected graph (without edge weights). Let $H$ be the edge-weighted graph obtained from $G$ as follows:

- Add two new vertices $s$ and $t$.
- Add edges from $s$ and $t$ every other vertex (including each other).
- Assign weight 1 to the edge $st$ and weight 0 to every other edge.

The total weight of all edges in $H$ is 1. Thus, a Hamiltonian cycle in $H$ is heavy if and only if it contains the edge $st$. I claim that $H$ contains a heavy Hamiltonian cycle if and only if $G$ contains a Hamiltonian path.

$\Rightarrow$ First, suppose $G$ has a Hamiltonian path from vertex $u$ to vertex $v$. By adding the edges $vs$, $st$, and $tu$ to this path, we obtain a Hamiltonian cycle in $H$. Moreover, this Hamiltonian cycle is heavy, because it contains the edge $st$.

$\Leftarrow$ On the other hand, suppose $H$ has a heavy Hamiltonian cycle. This cycle must contain the edge $st$, and therefore must visit all the other vertices in $H$ contiguously. Thus, deleting vertices $s$ and $t$ and their incident edges from the cycle leaves a Hamiltonian path in $G$.

Given $G$, we can easily construct $H$ in polynomial time by brute force.

Solution (smartass): I’ll describe a polynomial-time a reduction from the standard Hamiltonian cycle problem. Let $G$ be an arbitrary graph (without edge weights). Let $H$ be the edge-weighted graph obtained from $G$ by assigning each edge weight 0. I claim that $H$ contains a heavy Hamiltonian cycle if and only if $G$ contains a Hamiltonian path.

$\Rightarrow$ Suppose $G$ has a Hamiltonian cycle $C$. The total weight of $C$ is at least half the total weight of all edges in $H$, because $0 \geq 0/2$. So $C$ is a heavy Hamiltonian cycle in $H$.

$\Leftarrow$ Suppose $H$ has a heavy Hamiltonian cycle $C$. By definition, $C$ is also a Hamiltonian cycle in $G$.

Given $G$, we can easily construct $H$ in polynomial time by brute force.