The following problems ask you to prove some “obvious” claims about recursively-defined string functions. In each case, we want a self-contained, step-by-step induction proof that builds on formal definitions and prior results, not on intuition. In particular, your proofs must refer to the formal recursive definitions of string length and concatenation:

\[
|w| := \begin{cases} 
0 & \text{if } w = \varepsilon \\
1 + |x| & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x 
\end{cases}
\]

\[
w \cdot z := \begin{cases} 
z & \text{if } w = \varepsilon \\
a \cdot (x \cdot z) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x 
\end{cases}
\]

You may freely use the following results, which are proved in the lecture notes:

**Lemma 1:** \( w \cdot \varepsilon = w \) for all strings \( w \).

**Lemma 2:** \( |w \cdot x| = |w| + |x| \) for all strings \( w \) and \( x \).

**Lemma 3:** \((w \cdot x) \cdot y = w \cdot (x \cdot y)\) for all strings \( w, x, \) and \( y \).

The **reversal** \( w^R \) of a string \( w \) is defined recursively as follows:

\[
w^R := \begin{cases} 
\varepsilon & \text{if } w = \varepsilon \\
x^R \cdot a & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x 
\end{cases}
\]

For example, \( \text{STRESSED}^R = \text{DESSERTS} \) and \( \text{WTF374}^R = 473FTW \).

1. Prove that \( |w^R| = |w| \) for every string \( w \).

**Solution (induction on \( w \)):**

Let \( w \) be an arbitrary string.

Assume for any string \( x \) where \( |x| < |w| \) that \( |x^R| = |x| \).

There are two cases to consider.

- If \( w = \varepsilon \), then
  \[
  |w^R| = |\varepsilon| = |w| 
  \]

- Otherwise, \( w = ax \) for some symbol \( a \) and some string \( x \), and therefore
  \[
  |w^R| = |x^R \cdot a| = |x^R| + |a| = |x| + 1 = |w| 
  \]

In both cases, we conclude that \( |w^R| = |w| \).
Here is the argument for line (⋆) in more detail:

- Because $w = ax$, the definition of $|·|$ implies that $|w| = |x| + 1$.
- It follows that $|x| < |w|$.
- Thus, the inductive hypothesis implies that $|x| = |x^R|$.

Note that we are applying the inductive hypothesis to $x$, not to $x^R$.

2. Prove that $(w \cdot z)^R = z^R \cdot w^R$ for all strings $w$ and $z$.

Solution (induction on $w$):
Let $w$ and $z$ be arbitrary strings.
Assume for any string $x$ where $|x| < |w|$ that $(x \cdot z)^R = x^R \cdot z^R$.
There are two cases to consider:

- If $w = \varepsilon$, then
  
  \[
  (w \cdot z)^R = z^R \\
  = z^R \cdot \varepsilon \\
  = z^R \cdot w^R
  \]
  by definition of $\cdot$
  by Lemma 1
  by definition of $R$

- Otherwise, $w = ax$ for some symbol $a$ and some string $x$.
  
  \[
  (w \cdot z)^R = (a \cdot (x \cdot z))^R \\
  = (x \cdot z)^R \cdot a \\
  = (z^R \cdot x^R) \cdot a \\
  = z^R \cdot (x^R \cdot a) \\
  = z^R \cdot w^R
  \]
  by definition of $\cdot$
  by definition of $R$
  by the induction hypothesis (⋆)
  by Lemma 3
  by definition of $R$

In both cases, we conclude that $(w \cdot z)^R = z^R \cdot w^R$. ■

Again, in line (⋆) we are applying the induction hypothesis to $x$, which is legal because
the definition of $|·|$ implies $|x| < |w|$.

But how did I know that the induction hypothesis needs to change the first string $w$,
but not the second string $z$? I actually wrote down the inductive argument first, and
then noticed that in the proof for $w \cdot z$, I needed to argue inductively about $x \cdot z$. Same
string $z$, but $w$ changed to $x$.

Alternatively, as I mentioned in class on Tuesday, I could have noticed that the
recursive definition of $w \cdot z$ recurses on $w$ but leaves $z$ unchanged all the way down
to the base case. And inductive proofs always mirror the recursive definitions of the objects
in question, so . . .

Alternatively, in light of Lemma 2, we could have used induction on the sum of the
string lengths. Then the inductive hypothesis would read ‘Assume for all strings $x$ and $y$
such that $|x| + |y| < |w| + |z|$ that $(x \cdot y)^R = x^R \cdot y^R$.”
3. Prove that \((w^R)^R = w\) for every string \(w\).

**Solution (induction on \(w\)):**

Let \(w\) be an arbitrary string.

Assume for any string \(x\) where \(|x| < |w|\) that \((x^R)^R = x\).

There are two cases to consider.

- If \(w = \epsilon\), then \((w^R)^R = \epsilon^R = \epsilon\) by definition.
- Otherwise, \(w = ax\) for some symbol \(a\) and some string \(x\).

\[
\begin{align*}
(w^R)^R &= (x^R \cdot a)^R & \text{by definition of } R \\
&= a^R \cdot (x^R)^R & \text{by problem 2} \\
&= a \cdot (x^R)^R & \text{by definition of } R \\
&= a \cdot (x^R)^R & \text{by definition of } \cdot \\
&= a \cdot x & \text{by the induction hypothesis} \\
&= w & \text{by assumption}
\end{align*}
\]

In both cases, we conclude that \((w^R)^R = w\). ■
To think about later: Let \( #(a, w) \) denote the number of times symbol \( a \) appears in string \( w \). For example, \( #(X, \text{WTF374}) = 0 \) and \( #(0,000010101010010100) = 12 \).

4. Give a formal recursive definition of \( #(a, w) \).

Solution:

\[
#(a, w) = \begin{cases} 
0 & \text{if } w = \epsilon \\
1 + #(a, x) & \text{if } w = ax \text{ for some string } x \\
#(a, x) & \text{if } w = bx \text{ for some symbol } b \neq a \text{ and some string } x
\end{cases}
\]

5. Prove that \( #(a, w \cdot z) = #(a, w) + #(a, z) \) for all symbols \( a \) and all strings \( w \) and \( z \).

Solution (induction on \( w \)):

Let \( a \) be an arbitrary symbol, and let \( w \) and \( z \) be arbitrary strings.

Assume for any string \( x \) such that \( |x| < |w| \) that \( #(a, x \cdot z) = #(a, x) + #(a, z) \).

There are three cases to consider.

- If \( w = \epsilon \), then

\[
#(a, w \cdot x) = #(a, x) 
\]

by definition of \( \cdot \)

\[
= #(a, w) + #(a, x) 
\]

by definition of \( # \)

- If \( w = ax \) for some string \( x \), then

\[
#(a, w \cdot z) = #(a, ax \cdot z) 
\]

by definition of \( \cdot \)

\[
= #(a, a \cdot (x \cdot z)) 
\]

by definition of \( \cdot \)

\[
= 1 + #(a, x \cdot z) 
\]

by definition of \( # \)

\[
= 1 + #(a, x) + #(a, z) 
\]

by the induction hypothesis

\[
= #(a, ax) + #(a, z) 
\]

by definition of \( # \)

\[
= #(a, w) + #(a, z) 
\]

because \( w = ax \)

- If \( w = bx \) for some symbol \( b \neq a \) and some string \( x \), then

\[
#(a, w \cdot z) = #(a, b \cdot (x \cdot z)) 
\]

by definition of \( \cdot \)

\[
= #(a, x \cdot z) 
\]

by definition of \( # \)

\[
= #(a, x) + #(a, z) 
\]

by the induction hypothesis

\[
= #(a, bx) + #(a, z) 
\]

by definition of \( # \)

\[
= #(a, w) + #(a, z) 
\]

because \( w = bx \)

In every case, we conclude that \( #(a, w \cdot z) = #(a, w) + #(a, z) \).
6. Prove that \( \#(a, w^R) = \#(a, w) \) for all symbols \( a \) and all strings \( w \).

**Solution (induction on \( w \)):** Let \( a \) be an arbitrary symbol, and let \( w \) be an arbitrary string.

Assume for any string \( x \) such that \( |x| < |w| \) that \( \#(a, x^R) = \#(a, x) \).

There are three cases to consider.

- If \( w = \epsilon \), then \( w^R = \epsilon = w \) by definition, so \( \#(a, w^R) = \#(a, w) \).
- If \( w = ax \) for some string \( x \), then
  \[
  \#(a, w^R) = \#(a, x^R \cdot a) \quad \text{by definition of } R
  \]
  \[
  = \#(a, x^R) + \#(a, a) \quad \text{by problem 5}
  \]
  \[
  = \#(a, x^R) + 1 \quad \text{by definition of } \#
  \]
  \[
  = \#(a, x) + 1 \quad \text{by the induction hypothesis}
  \]
  \[
  = \#(a, w) \quad \text{by definition of } \#
  \]
- If \( w = bx \) for some symbol \( b \neq a \) and some string \( x \), then
  \[
  \#(a, w^R) = \#(a, x^R \cdot b) \quad \text{by definition of } R
  \]
  \[
  = \#(a, x^R) + \#(a, b) \quad \text{by problem 5}
  \]
  \[
  = \#(a, x^R) \quad \text{by definition of } \#
  \]
  \[
  = \#(a, x) \quad \text{by the induction hypothesis}
  \]
  \[
  = \#(a, w) \quad \text{by definition of } \#
  \]

In every case, we conclude that \( \#(a, w^R) = \#(a, w) \). \( \blacksquare \)