1. For any integer k, the problem kSAT is defined as follows:

   - INPUT: A boolean formula Φ in conjunctive normal form, with exactly k distinct literals in each clause.
   - OUTPUT: TRUE if Φ has a satisfying assignment, and FALSE otherwise.

   (a) Describe a polynomial-time reduction from **2**SAT to **3**SAT, and prove that your reduction is correct.

   Solution: Let Φ be an arbitrary 2CNF boolean formula. We construct a 3CNF formula Φ' by splitting each clause $(a \vee b)$ in Φ into two clauses $(a \vee b \vee x) \wedge (a \vee b \vee \bar{x})$, where x is a new variable. I claim that Φ is satisfiable if and only if Φ' is satisfiable.

   - Suppose Φ is satisfiable. Fix an arbitrary satisfying assignment of Φ. Consider an arbitrary clause $(a \vee b)$ in Φ; at least one of the literals a or b must be true. Thus, no matter what value we assign to the new variable x, the new clauses $(a \vee b \vee x) \wedge (a \vee b \vee \bar{x})$ in Φ' each contain at least one true literal. We conclude that Φ' is satisfiable.
   - Suppose Φ' is satisfiable. Fix an arbitrary satisfying assignment of Φ'. Consider an arbitrary clause $(a \vee b)$ in Φ. The corresponding pair of clauses $(a \vee b \vee x) \wedge (a \vee b \vee \bar{x})$ in Φ' each contain at least one true literal. No matter what value x has, at least one of the literals a or b must be true, so the clause $(a \vee b)$ is satisfied. We conclude that Φ is satisfiable.

   Transforming Φ into Φ' by brute force takes linear time.

   Essentially the same algorithm reduces kSAT to $(k+1)$SAT for any k. ■

   Solution (smartass): See part (b). We don't need to call the 3SAT algorithm. ■

   Rubric: 5 points: standard poly-time reduction rubric (scaled). Yes, the smartass solution is worth full credit.

   (b) Describe and analyze a polynomial-time algorithm for **2**SAT. *[Hint: This problem is strongly connected to topics covered earlier in the semester.]*

   Solution: Let Φ be an arbitrary 2CNF boolean formula. Suppose Φ has n variables and k clauses. We define a directed graph $G = (V, E)$ as follows:

   - V contains $2n$ vertices, which correspond to the possible literals in Φ. Each literal becomes a single vertex, even if it appears in multiple clauses or no clauses (like the 3COLOR reduction shown in class, but not the INDEPENDENTSET reduction).
   - E contains $2k$ edges: $\bar{a} \to b$ and $\bar{b} \to a$ for each clause $(a \vee b)$ in Φ. Thus, the edges correspond to logical implications.

   Once we construct the graph G, we compute its strong components using either of the algorithms in the lecture notes. Finally, return TRUE if no variable vertex x is in the same strong component as its negation $\bar{x}$; otherwise, return FALSE. The entire algorithm runs in $O(V + E) = O(n + k)$ time.

With a bit more work, we can actually construct a satisfying assignment if one exists. Build the strong component graph of G, topologically order the strong components, and label each vertex by the position of its strong component in this topological order. For each variable v, there are three cases to consider.

- If $label(x) < label(\bar{x})$, set $x \leftarrow$ FALSE.
- If $label(x) > label(\bar{x})$, set $x \leftarrow$ TRUE.
- If $label(x) = label(\bar{x})$, then x and $\bar{x}$ are in the same strong component, so there is no satisfying assignment. Abort immediately.

We can prove the algorithm correct as follows.

- Suppose no variable is strongly connected with its negation. Then we successfully assign a value to each variable. If some clause $(a \vee b)$ in Φ has no true literals, then we have a contradiction:

$$\begin{aligned} label(a) &< label(\bar{a}) && \text{because } a = \text{FALSE} \\ &\le label(b) && \text{because } \bar{a} \rightarrow b \in E \\ &< label(\bar{b}) && \text{because } b = \text{FALSE} \\ &\le label(a) && \text{because } \bar{b} \rightarrow a \in E \end{aligned}$$

  We conclude that every clause in Φ is satisfied, as claimed.
- On the other hand, if we find a variable and its negation in the same strong component, then no matter which value we assume for that variable, we can derive a contradiction, which implies that Φ has no satisfying assignment.

The entire algorithm runs in $O(V + E) = O(n + k)$ time. ■

Rubric: 4 points: standard graph-reduction rubric

(c) Why don't these results imply a polynomial-time algorithm for 3SAT?

Solution: The reduction in part (a) goes in the wrong direction. If we want to prove that 2SAT is NP-hard, or that 3SAT can be solved in polynomial time, we need to describe a reduction *from 3SAT to 2SAT*. ■

Rubric: 1 point; all or nothing.

2. (a) Describe a polynomial-time reduction from SubsetSum to Partition.

Solution (add two elements): Let x denote the sum of the elements of X.

- If $k > x$, then $\textsc{SubsetSum}(X, k) = \textsc{False}$.
- Otherwise, if $k = x/2$, then $\textsc{SubsetSum}(X, k) = \textsc{Partition}(X)$.
- Otherwise, $\textsc{SubsetSum}(X, k) = \textsc{Partition}(X \cup \{2x + k, 3x - k\})$.

The reduction is trivially correct when $k > x$, and the proof of correctness when $k = x/2$ is the same as part (b), so let's assume that $k \le x$ and $k \ne x/2$. Let $Y = X \cup \{3x - k, 2x + k\}$. Observe that $\sum Y = 6x$ and $2x + k \ne 3x - k$.

$\Longrightarrow$ Suppose X has a subset X' whose elements sum to k. Let $Y_1 = X' \cup \{3x - k\}$, and let $Y_2 = Y \setminus Y_1$. and $\sum Y_1 = \sum X + 3x - k = 3x$, so $\sum Y_2 = 3x$ as well. So Y can be partitioned into two subsets with the same sum.

$\Longleftarrow$ On the other hand, suppose Y can be partitioned into two subsets Y_1 and Y_2 with the same sum, which must be $3x$. Neither set can contain both $2x + k$ and $3x - k$, because $(2x + k) + (3x - k) = 3x > x$. So suppose without loss of generality that $3x - k \in Y_1$, and let $X' = Y_1 \setminus \{3x - k\}$. Then X' is a subset of X whose elements sum to k.

The reduction takes $O(n)$ time (to compute x). ■

Solution (add one element): Let x denote the sum of the elements of X.

- If $k = x/2$, then clearly $\textsc{SubsetSum}(X, k) = \textsc{Partition}(X)$.
- Otherwise, $\textsc{SubsetSum}(X, k) = \textsc{Partition}(X \cup \{|x - 2k|\})$.

The reduction is trivially correct when $k > x$, and the proof of correctness when $k = x/2$ is the same as part (b), so let's assume that $k \le x$ and $k \ne x/2$.

Let $Y = X \cup \{|x - 2k|\}$, and let $y = \sum Y$. If $x > 2k$, we have $y = 2x - 2k$; otherwise, we have $y = 2k$.

$\Longrightarrow$ Suppose X has a subset X' whose elements sum to k. If $x > 2k$, let $Y' = X' \cup \{x - 2k\}$; in this case, we have $\sum Y' = x - k = y/2$. On the other hand, if $x < 2k$, let $Y' = X'$; in this case, we have $\sum Y' = k = y/2$. In either case Y can be partitioned into two subsets (Y' and $Y \setminus Y'$) with the same sum.

$\Longleftarrow$ On the other hand, suppose Y can be partitioned into two subsets Y_1 and Y_2 with the same sum. Without loss of generality, suppose $|x - 2k| \in Y_1$. If $x > 2k$, let $X' = Y_1 \setminus \{x - 2k\}$; otherwise, let $X' = Y_2$. In both cases, X' is a subset of X whose elements sum to k.

The reduction takes $O(n)$ time (to compute x). ■

Rubric: 5 points, standard poly-time reduction rubric (scaled). These are not the only correct reductions. No penalty if the output of the reduction could be a multiset instead of a set. —½ if the output of the reduction can contain negative integers.

(b) Describe a polynomial-time reduction from PARTITION to SUBSETSUM.

Solution: $\textsc{Partition}(Y) = \textsc{SubsetSum}(Y, y/2)$, where y is the sum of the elements of Y.

$\Longrightarrow$ Suppose Y can be partitioned into two subsets Y_1 and Y_2 with the same sum. Then $\sum Y_1 = \sum Y_2 = y/2$. So Y has a subset whose elements sum to $y/2$.

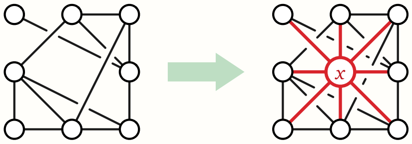
$\Longleftarrow$ On the other hand, suppose Y has a subset Z whose elements sum to $y/2$. Then the elements of $Y \setminus Z$ sum to $y/2$ as well, which implies that $\sum Z = \sum (Y \setminus Z)$. So Y can be partitioned into two subsets with the same sum.

The reduction requires $O(n)$ time (to compute y). ■

Rubric: 5 points standard reduction rubric (scaled). This is not the only correct reduction.

3. *Pebbling* is a solitaire game played on an undirected graph G, where each vertex has zero or more *pebbles*. A single *pebbling move* removes two pebbles from some vertex v and adds one pebble to an arbitrary neighbor of v. (Obviously, v must have at least two pebbles before the move.) The PEBBLECLEARING problem asks, given a graph $G = (V, E)$ and a pebble count $p(v)$ for each vertex v, whether is there a sequence of pebbling moves that removes all but one pebble. Prove that PEBBLECLEARING is NP-hard.

Solution: We describe a reduction from the undirected Hamiltonian path problem. Let $G = (V, E)$ be an arbitrary undirected graph, and suppose G has n vertices. We construct a new graph H by adding a new vertex x to G, and then adding edges from x to every vertex of G. (Vertex x is sometimes called an *apex*.) Finally, we place two pebbles on x and one pebble on every other vertex.



I claim that G has a Hamiltonian path if and only if we can clear all but one of the pebbles from H.

$\Longrightarrow$ Suppose G has a Hamiltonian path $v_1 \to v_2 \to \cdots v_n$. Then we can clear all but one of the pebbles from H by making pebble moves from x to v_1, then from v_i to v_{i+1} for all i in increasing order, and finally from v_n back to v_{n-1}.

$\Longleftarrow$ Suppose we can clear all the pebbles from H. Fix a sequence of pebble moves. We start with $n + 1$ pebbles and end with one pebble, so the sequence must contain exactly $n + 1$ pebble moves. For each index i, let $u_i \to v_i$ denote the ith pebble move, which removes two pebbles from u_i and adds one pebble to v_i. The definition of pebble moves implies that $u_i v_i$ is an edge in H, for every index i. Every vertex in H contains at least one pebble, so there must be at least one pebble move away from each vertex; it follows that the vertices u_i are distinct.

Because x is the only vertex with two pebbles initially, we have $u_1 = x$. Now I need to prove the following claim:

Claim: *For every index $1 \le i \le n$, we have $v_i = u_{i+1}$; moreover, the following conditions hold just after the ith pebble move: $u_1, \ldots, u_i$ have no pebbles; u_{i+1} has two pebbles; and every other vertex has one pebble.*

Proof: Fix an index $1 \le i \le n$, and consider the placement of pebbles just *before* the ith move $u_i \to v_i$. By the induction hypothesis, $u_1, \ldots, u_{i-1}$ are empty, u_i has two pebbles, and all other vertices have one pebble each. We must have $v_i = u_j$ for some $j > i$, since otherwise no more moves would be available. Thus, after the ith move, vertices $u_1, \ldots, u_i$ are empty, vertex v_i has two pebbles, and all other vertices have one pebble each. Because v_i is the only vertex with more than one pebble, we must have $u_{i+1} = v_i$. □

We conclude that the sequence $u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_{n+1}$ is a walk in H that visits each vertex exactly once, or in other words, a Hamiltonian path in H. Removing the first vertex $x = u_1$ gives us a Hamiltonian path in the original graph G.

Transforming the original graph G into the pebbled graph H obviously takes only $O(V)$ time. ■

Rubric: 10 points: standard polynomial-time reduction rubric. This is not the only correct solution; in particular, there is also a (many-one) reduction from undirected Hamiltonian cycle. This is more detail than necessary for full credit.