Rice's Theorem

Let $\mathcal{L}$ be any family of languages such that

- There is a machine Y s.t $\text{ACCEPT}(Y) \in \mathcal{L}$ ←
- There is a machine N s.t $\text{ACCEPT}(N) \notin \mathcal{L}$

The language $\{\langle M \rangle \mid \text{ACCEPT}(M) \in \mathcal{L}\}$ is undecidable.

One of these works for Y or N

"return TRUE"   "Return FALSE"

Proof: Fix $\mathcal{L}$ Assume $\boxed{\text{wlog } \emptyset \notin \mathcal{L}}$ Fix Y

$N =$ "return FALSE"

Suppose machine ~~A~~ that decides $\{\langle M \rangle \mid \text{Accept}(M) \in \mathcal{L}\}$

Build algorithm to decide halting problem:

$H(\langle M \rangle, w)$:

> Write the following encoding:
>
> $WTF(x)$:
>
> > run M on input w
> > run Y on input x
>
> return $A(\langle WTF \rangle)$

Suppose M halts on w

$\Rightarrow$ WTF is Y

$\Rightarrow$ ACCEPT(WTF) = ACCEPT(Y) $\in \mathcal{L}$

$\Rightarrow$ A accepts <WTF>

$\Rightarrow$ H accepts <M>, w

Suppose M hangs on w

$\Rightarrow$ WTF hangs on <u>all</u> inputs

$\Rightarrow$ ACCEPT(WTF) = $\emptyset \notin \mathcal{L}$

$\Rightarrow$ A rejects <WTF>

$\Rightarrow$ H rejects <M>, w

$\Pi = \{0^n \mid 0^n$ appears in decimal of $\pi\}$

<u>AtMostK(0^n):</u>

return $n \le k$

return True

Universal Models Turing-complete

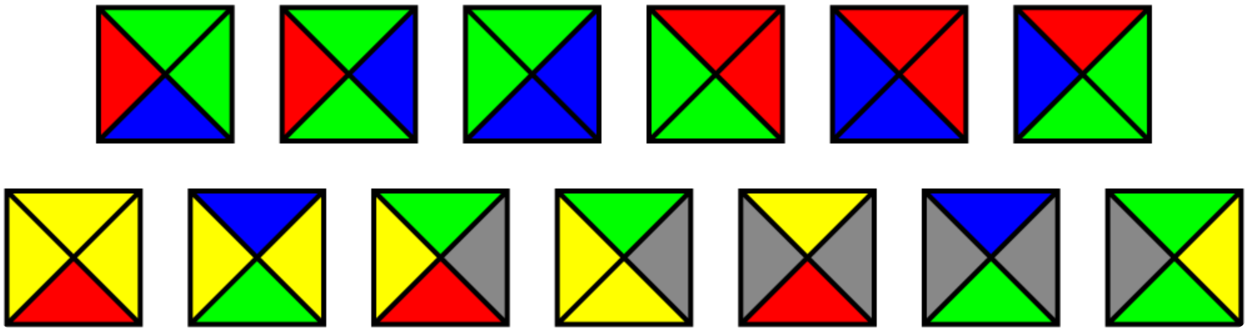
<u>Universal TM</u> = TM interpreter written in TM

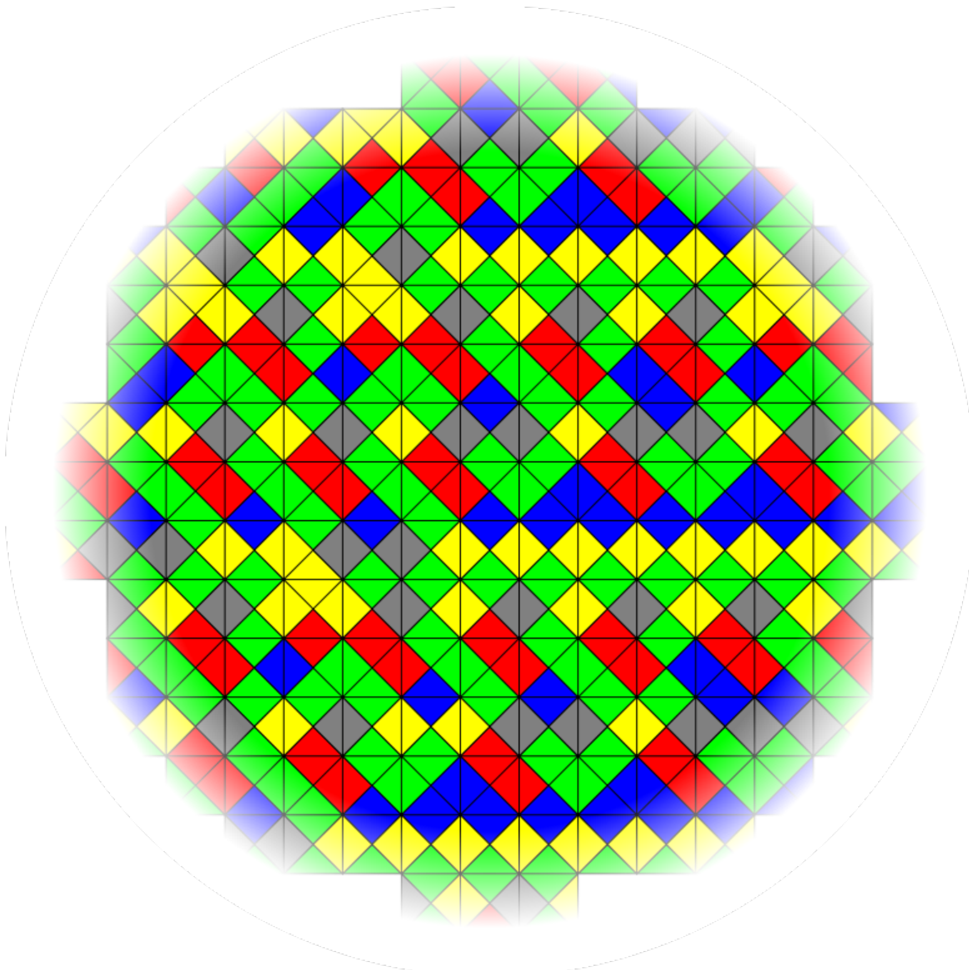
5 states × 4 symbols

2 states 7

8 2

Wang tiles





5/19,
398483185/122130
83,
185/182283780455
4996917562403854
272365/67, 43/5,
43/71, 125173/47,
54298917151/9528
9508861012466225
1213649,
89139597/13,
8752951/23, 17/43,
5/17, 31/53,
1829/41,
59/73, 331639/23,
3713/31, 79/83,
31/79,
8633/7, 101/97,
9797/13, 9797/47,
9167/13, 103/107,

1558654261983
1329049680170

1761449,
655/41,

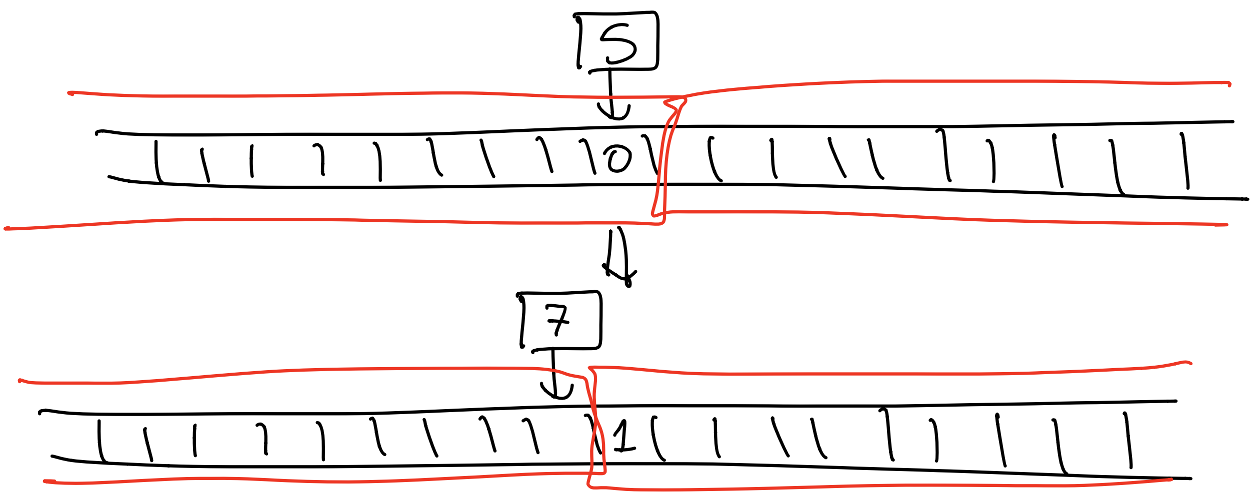
1459150059235
09757913927,
07133/4142651
1605851507159

17/29, 6409/47,
17042839/7,

4307/41, 89/59,
268837/23,

68579/11,
35/101,

1774381/47, 109/103, 109/113, 578899/23, 11227/13, 127/109, 127/131,
16637/47, 16637/11, 1114679/61, 2/127, 5/2, 3/37



2 stacks

push
pop

$\cdots \cancel{000} 0 \mid 11\ 1000\ 10100$

$\boxed{5}$

$\cdots \cancel{000} 0 \mid 1\ 1\ 101\ 110101$

2 integers x y

$x \leftarrow 2x$
$x \leftarrow 2x+1$
$x \leftarrow \lfloor x/2 \rfloor$

y

$\boxed{\text{goto}}$

if x odd

counters a b c

$a = x$ $b = y$ $c = 0$

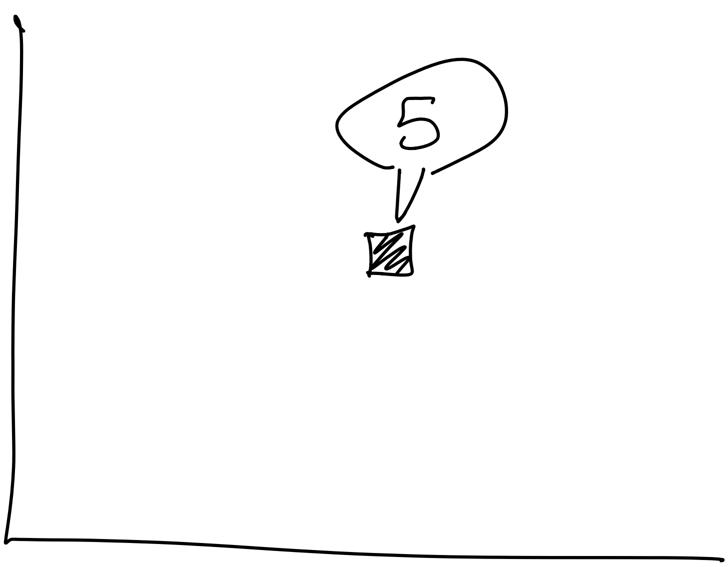
a++
a--
if $a == 0$

goto

while $a > 0$
  a--
  a--
  c++
while $c > 0$
  a++
  c--

$x \leftarrow x/2$

2 counters $p = 2^a 3^b 5^c$ $q = 0$



Karel the Robot

FRACTRAN

$$\frac{3}{2}, \frac{5}{3}, \frac{6}{5}, 1$$

4 6 9 15 25 30 45 75 . . . .

Collatz

$\frac{1}{2}$, 3 (+1)

```
// CLF-INTERPRET: FRACTRAN interpreter written in FRACTRAN
//
// Chris Lomont, Apr 2017
//
// Input format:
//   The start state for the interpreted program is in variable s
//   The encoded program to be interpreted is in variable p
//   The program is encoded into p as follows
// 1. 0 pad each num and den on left to make same length (per fraction)
// 2. Expand all integers base 10
// 3. Alternate digits, numerator first, then denominator, etc.
// 4. for each fraction prepend  0, append 10
// 5. at end of program, append 10 to signal end of fractions
// 6. Encode result as a base 11 number to get p, with
//    first terms in sequence as least sig digits (i.e., reverse list)
//
```

```
//    {digits2[{21/3, 4/17}], encode2[{21/3, 4/17}]}
//    {{0, 7, 1, 10, 0, 0, 1, 4, 7, 10, 10}, 284533968840}
//
//
// Execute CLF-INTERPRET with start state 5 * 7^s * 67^p
// Whenever CLF-INTERPRET state is divisible by 2, the exponent of the prime
// factor of 7 holds the interpreted state. When the CLF-INTERPRET state is
// divisible by 3, the interpreted program has halted. CLF-INTERPRET will
// halt soon thereafter.
//
// The program is 48 fractions using 32 unique primes.
//

Factored form:

5/19, 61^10*23*19/67^11*5, 37/67^10*5, 47^10*19/41*5, 61*47*19/67*5,
43/5, 43/71, 41*71/47*43, 61^10*67*71/23^11*43, 61^10*31/23^10*43,
47^10*71/13*43, 61*47*71/23*43, 17/43, 17/29, 13*29/47*17, 5/17,
31/53, 11*23*53/7*41*31, 59/41*31, 59/73, 7*73/23*11*59, 73/41*59,
89/59, 47*79/31, 79/83, 41*83/23*79, 31/79, 97/7*89, 101/97,
7*97/11*101, 97/13*101, 97/47*101, 7*5/101, 103/13*89, 103/107,
7*23*107/47*103, 109/103, 109/113, 47*113/23*109, 103/13*109, 127/109,
127/131, 131/47*127, 131/11*127, 67*131/61*127, 2/127, 5/2,3/37

Expanded form:

5/19, 1558654261983398483185/122130132904968017083,
185/1822837804551761449, 4996917562403854655/41, 272365/67, 43/5,
43/71, 125173/47, 145915005923554298917151/952809757913927,
950886101246622507133/41426511213649, 160585150715989139597/13,
8752951/23, 17/43, 17/29, 6409/47, 5/17, 31/53, 17042839/7, 1829/41,
59/73, 331639/23, 4307/41, 89/59, 3713/31, 79/83, 268837/23, 31/79,
8633/7, 101/97, 68579/11, 9797/13, 9797/47, 35/101, 9167/13, 103/107,
1774381/47, 109/103, 109/113, 578899/23, 11227/13, 127/109, 127/131,
16637/47, 16637/11, 1114679/61, 2/127, 5/2, 3/37
```