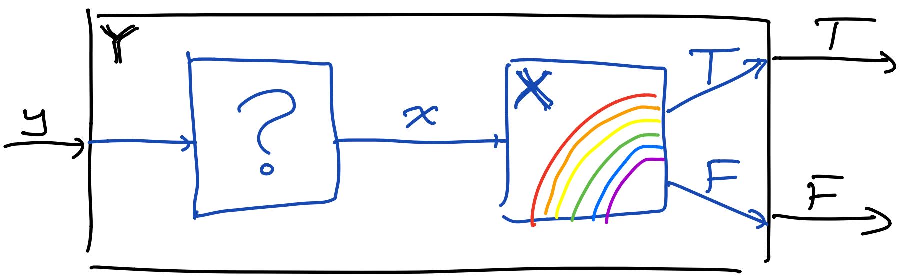
NP-hardness (again, for the last time)

NP-hard = no polynomial-time algorithm (morally)

To prove X is NP-hard:

Reduce known NP-hard problem Y to X:

- Describe poly-time algorithm to transform arbitrary instance y of Y to special instance x of X so that
  - If y is a "good" instance of Y, then x is a "good" instance of X
  - If y is a "bad" instance of Y then x is a "bad" instance of X



One transformation

Two proofs

satisfying assignment
large independent set
3-coloring
Ham. cycle, ...

Typical proof:

- Show that any certificate that y is good for Y becomes a certificate that x is good for X
- Show that any certificate that x is good for X must come from a certificate that y is good for Y

This is the hard part!

Reduction must force structure on certificates for X

COMPUTER **SCIENCE** **Questions** Tags Users

Pebbling Problem

Gee, this looks Familiar

Pebbling is a solitaire game played on an undirected graph G, where each vertex has zero or more pebbles. A single pebbling move consists of removing two pebbles from a vertex v and adding one pebble to an arbitrary neighbor of v. (Obviously, the vertex v must have at least two pebbles before the move.) The PebbleDestruction problem asks, given a graph $G = (V; E)$ and a pebble count $p(v)$ for each vertex v, whether there is a sequence of pebbling moves that removes all but one pebble. Prove that PebbleDestruction is NP-complete.

10

2

First, I show that it is in NP since I can verify the solution in polynomial time, tracing back the pebble count from just one pebble.

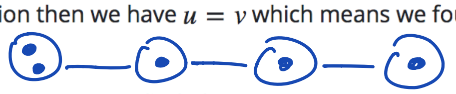
Next, what are some ideas on which problems to use as the basis for a polynomial-time reduction?

1 Answer

active oldest **votes**

8

Suppose in a graph G there is one pebble on each vertex except one vertex v with $p(v) = 2$, then above pebbling problem has solution on G *iff* G has a Hamiltonian circuit. It's easy to check if there is a Hamiltonian circuit, then there is a solution for pebbling on G. On the other hand, in any solution to the pebbling, we should start from vertex v. Suppose that we visit some vertex u twice such that this u is the first vertex which visited twice in G by pebbling algorithm, then we have a loop which starts from u and ends in u and finally because u is the first for making loop then we have $p(u) = 1$ so we cannot continue pebbling algorithm. Indeed if the algorithm has a solution then we have $u = v$ which means we found a Hamiltonian circuit which starts in v.



share cite edit flag edited Jul 9 '13 at 16:23 answered Apr 28 '13 at 19:42 user742

This is incorrect! Missed the "only-if" proof!

Choosing what to reduce from:

- Finding large subset? Max Clique or Max Ind Set
- Finding small subset? Min Vertex Cover
- Finding subset with specific properties? SAT
- Labeling/classifying? Coloring
- Long sequence? Hamiltonian something
- Balancing/packing? (3) Partition
- 3 something? 3SAT/3Color/3Partition
- Give up? 3SAT / Circuit SAT

Some useful NP-hard problems. You are welcome to use any of these in your own NP-hardness proofs, except of course for the specific problem you are trying to prove NP-hard.

defaults

CircuitSat: Given a boolean circuit, are there any input values that make the circuit output True?

3Sat: Given a boolean formula in conjunctive normal form, with exactly three distinct literals per clause, does the formula have a satisfying assignment?

big subset

MaxIndependentSet: Given an undirected graph G, what is the size of the largest subset of vertices in G that have no edges among them?

MaxClique: Given an undirected graph G, what is the size of the largest complete subgraph of G?

small subset

MinVertexCover: Given an undirected graph G, what is the size of the smallest subset of vertices that touch every edge in G?

MinSetCover: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subcollection whose union is S?

MinHittingSet: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subset of S that intersects every subset S_i?

classifying

3Color: Given an undirected graph G, can its vertices be colored with three colors, so that every edge touches vertices with two different colors?

sequencing

HamiltonianPath: Given graph G (either directed or undirected), is there a path in G that visits every vertex exactly once?

HamiltonianCycle: Given a graph G (either directed or undirected), is there a cycle in G that visits every vertex exactly once?

TravelingSalesman: Given a graph G (either directed or undirected) with weighted edges, what is the minimum total weight of any Hamiltonian path/cycle in G?

LongestPath: Given a graph G (either directed or undirected, possibly with weighted edges), what is the length of the longest simple path in G?

connecting

SteinerTree: Given an undirected graph G with some of the vertices marked, what is the minimum number of edges in a subtree of G that contains every marked vertex?

balancing or packing

SubsetSum: Given a set X of positive integers and an integer k, does X have a subset whose elements sum to k?

Partition: Given a set X of positive integers, can X be partitioned into two subsets with the same sum?

3Partition: Given a set X of $3n$ positive integers, can X be partitioned into n three-element subsets, all with the same sum?

other

IntegerLinearProgramming: Given a matrix $A \in \mathbb{Z}^{n\times d}$ and two vectors $b \in \mathbb{Z}^n$ and $c \in Z^d$, compute $\max\{c \cdot x \mid Ax \leq b, x \geq 0, x \in \mathbb{Z}^d\}$.

FeasibleILP: Given a matrix $A \in \mathbb{Z}^{n\times d}$ and a vector $b \in \mathbb{Z}^n$, determine whether the set of feasible integer points $\max\{x \in \mathbb{Z}^d \mid Ax \leq b, x \geq 0\}$ is empty.

silly

Draughts: Given an $n \times n$ international draughts configuration, what is the largest number of pieces that can (and therefore must) be captured in a single move?

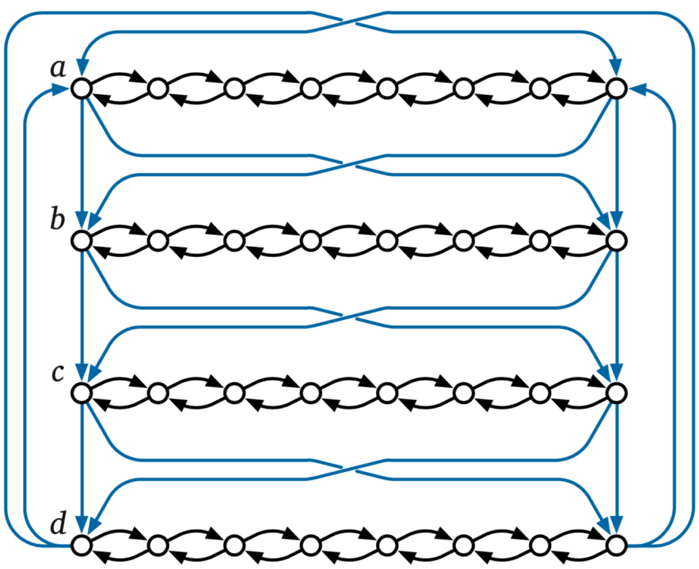
SuperMarioBrothers: Given an $n \times n$ Super Mario Brothers level, can Mario reach the castle?

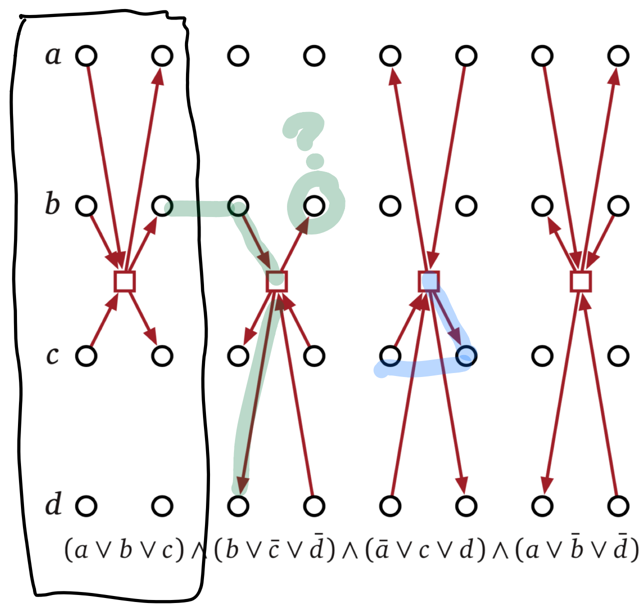
SteamedHams: Aurora borealis? At this time of year, at this time of day, in this part of the country, localized entirely within your kitchen? May I see it?

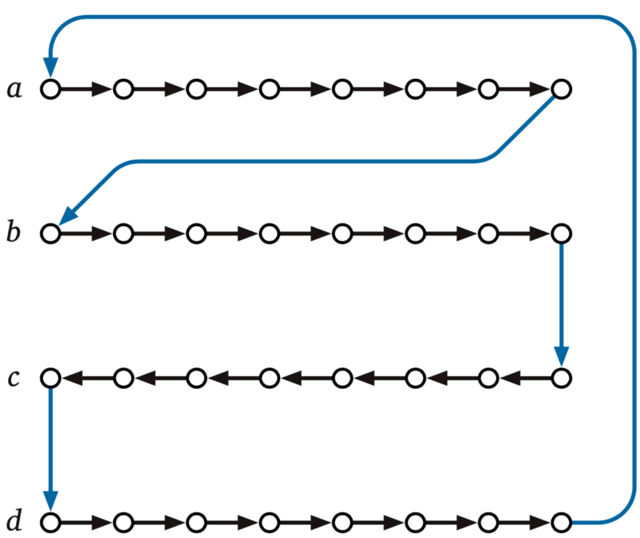
Reduce 3SAT to Ham Cycle

variable gadgets

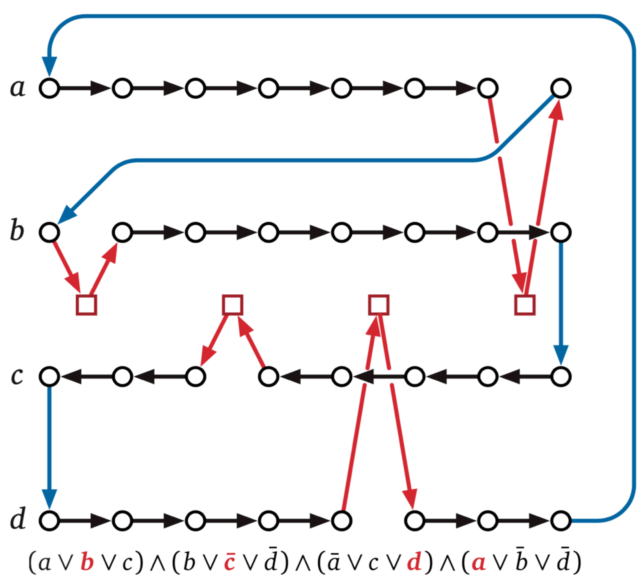
clause gadgets







$a = b = d = \text{TRUE},\ c = \text{FALSE}$



Classic Nintendo Games are (Computationally) Hard

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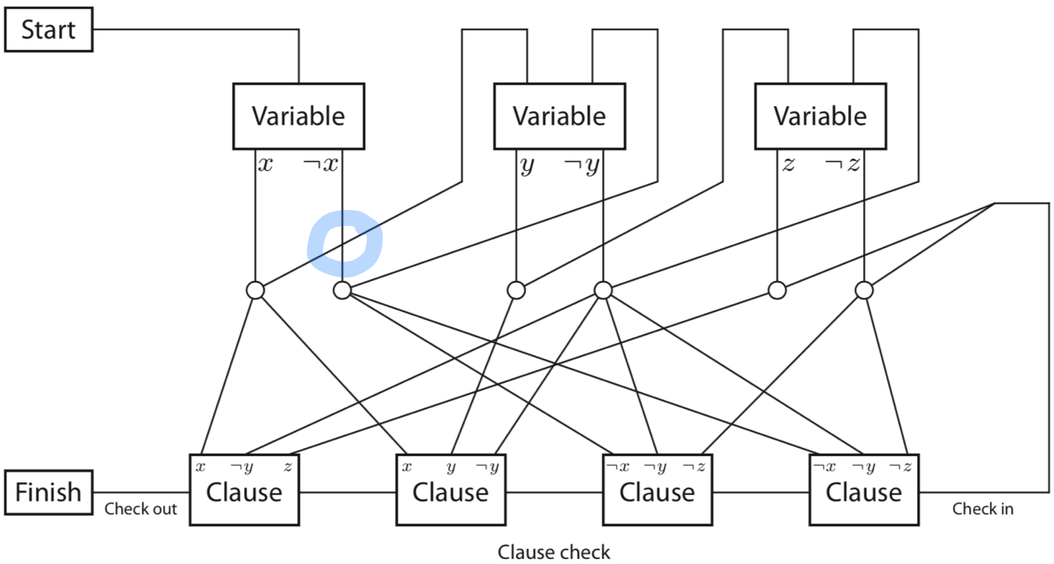
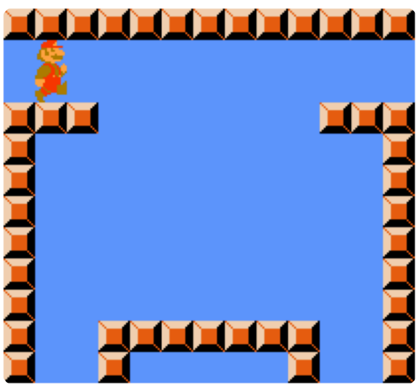
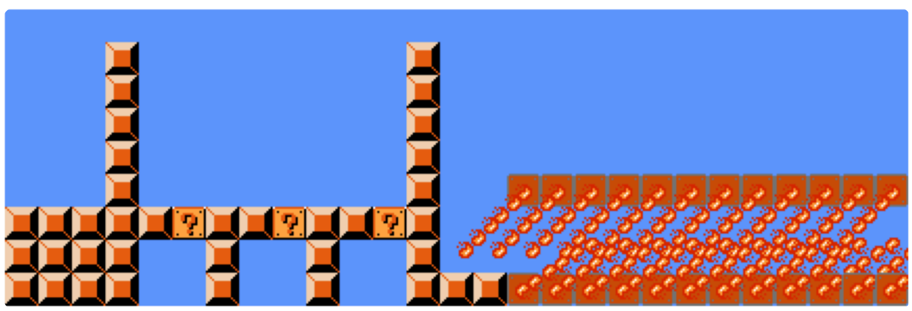
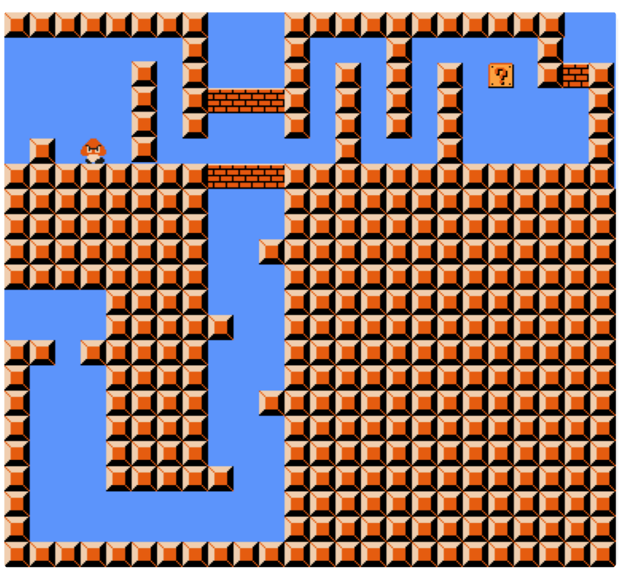
February 10, 2015









Solving the Rubik's Cube Optimally is NP-complete

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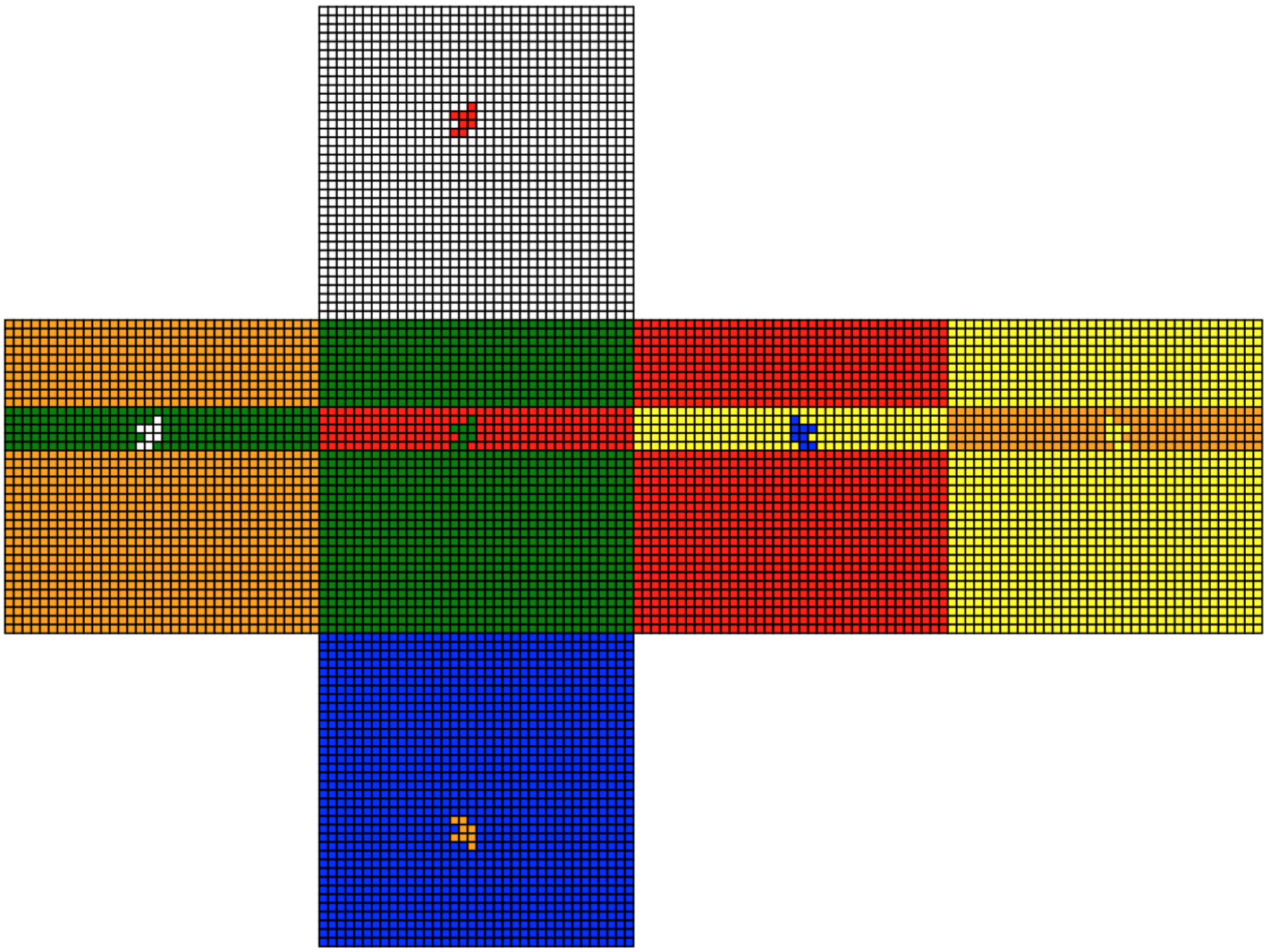
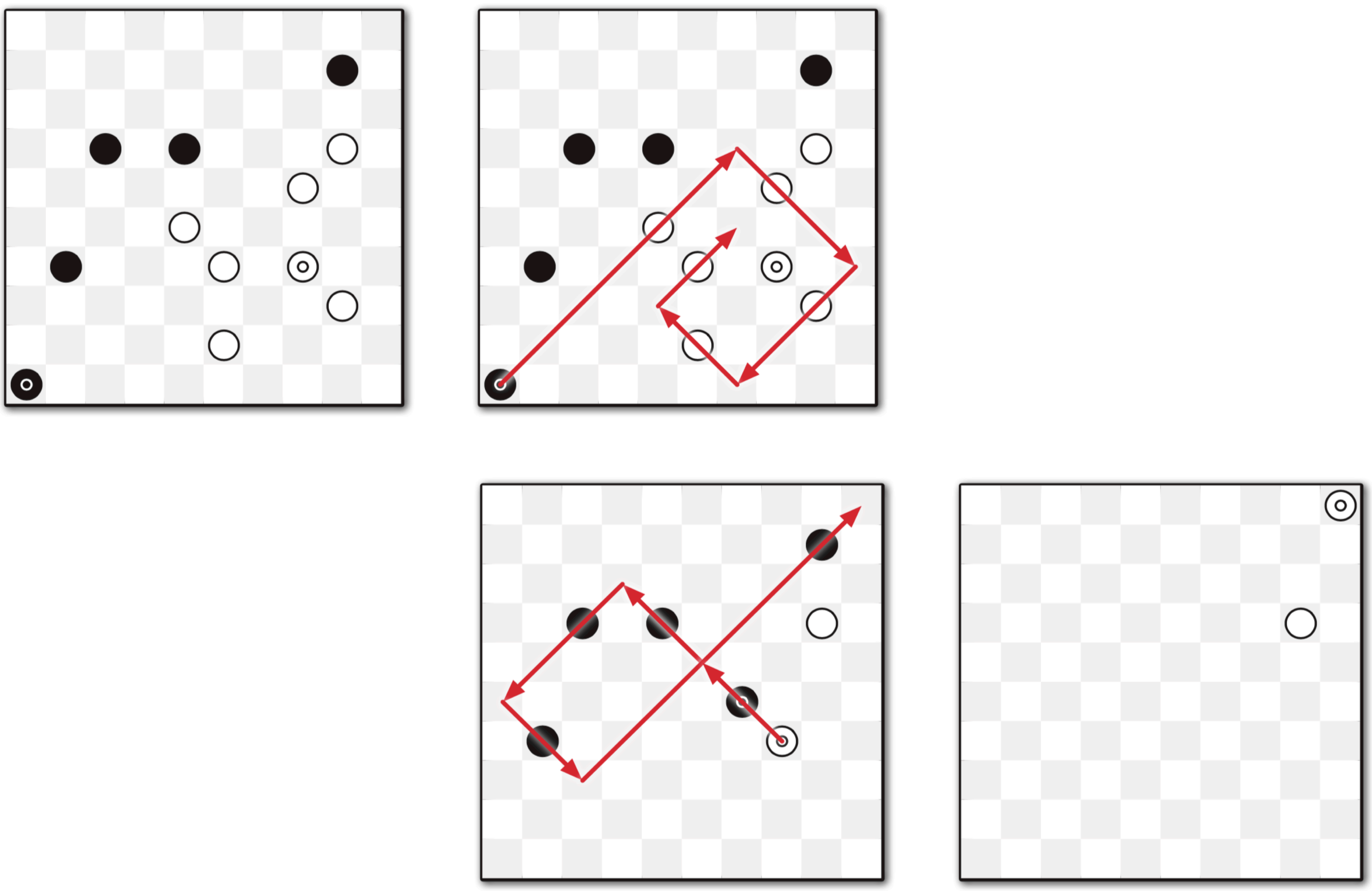
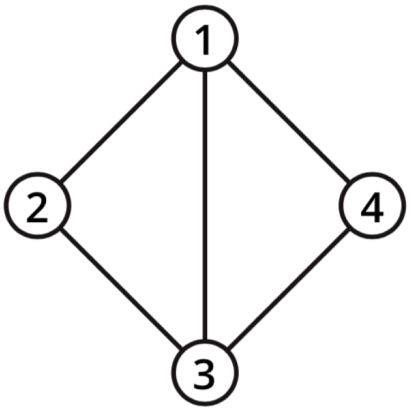


draughts

- Flying kings!
- Captured pieces stay to end of move
- Forced capture max # pieces

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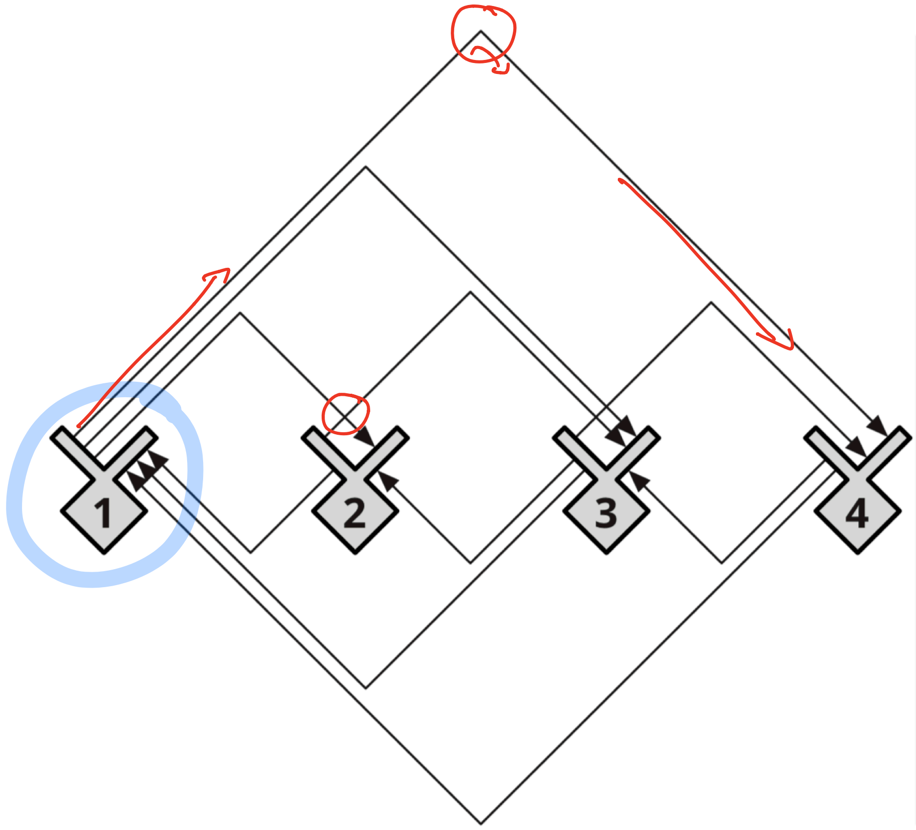
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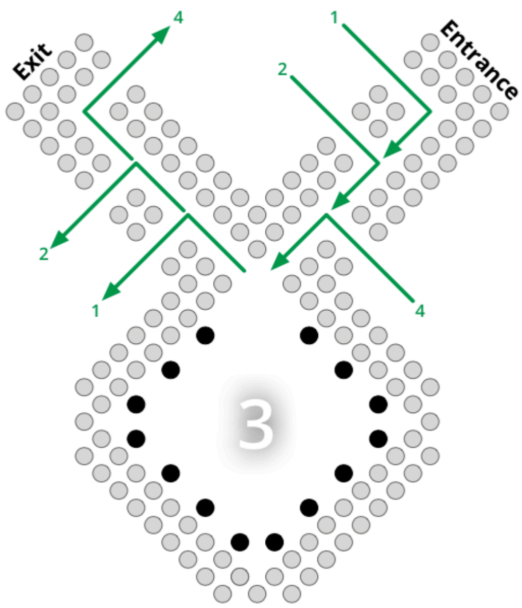
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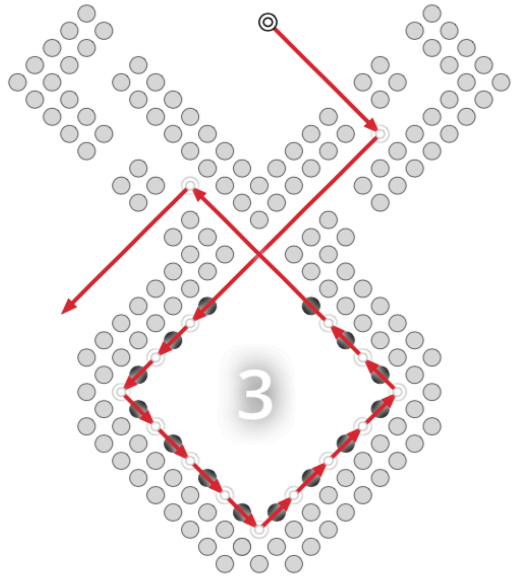
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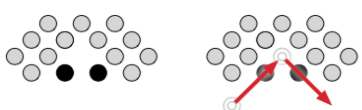
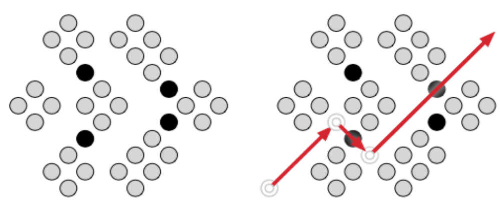

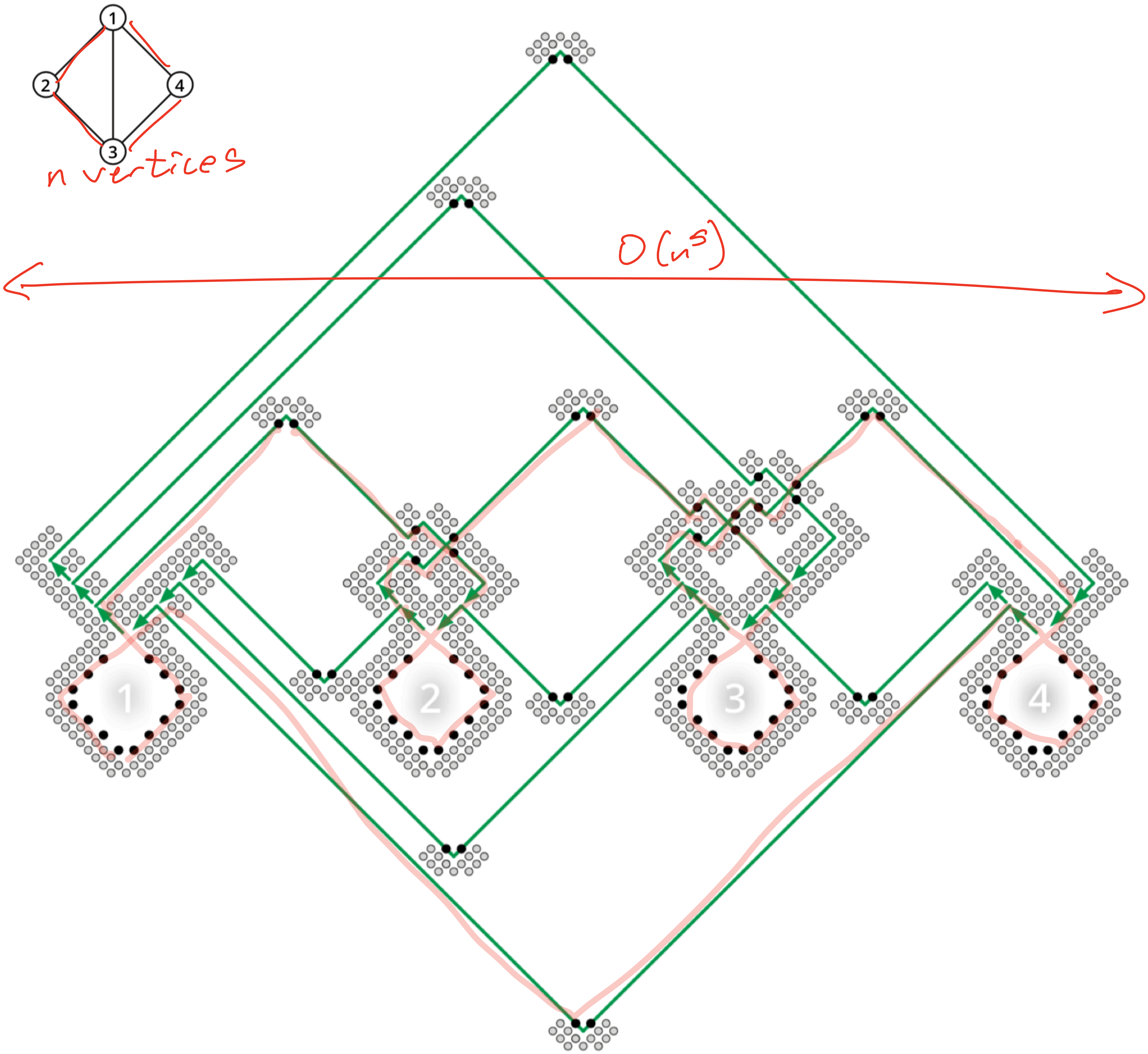

$O(n^2)$ edges

$\rightarrow O(n^4)$ crossing gadgets

So need $\gg n^3$ black pieces in each vertex gadget

use $\Theta(n^4)$

So width $= \Theta(n^5)$, area $= \Theta(n^{10})$

polynomial!