$P \quad \text{Solvable in polynomial time}$
$NP \quad \text{Checkable in polynomial time}$

$x \text{is NP-hard} \quad \text{If poly time algo for } x, \text{ then } P=NP$

Cook-Levin Theorem: CircuitSAT is NP-hard!

To prove $X$ is NP-hard
Reduce CircuitSAT to $X$ in poly time.
- or 3SAT
- or MaxIndSet/MaxClique/MinVertexCover
- or 3Color/4Color/MinColor/…

Transform arb. input $K$ to CircuitSAT int an input $x$ for $X$
- True outputs are correct
- False outputs are correct
Max Independent Set \textit{NP}-hard

\textbf{MaxIndependentSet}

\begin{align*}
G & \quad \text{graph} \\
\text{complement in } O(V^2) & \quad \text{time} \\
\tilde{G} & \quad \text{graph} \\
\text{MaxClique} & \quad k \\
& \quad \text{size of largest clique in } \tilde{G} \\
k & \quad \text{size of largest independent set in } \tilde{G}
\end{align*}

\textbf{MaxIndependentSet}

\begin{align*}
G & \quad \text{graph} \\
\text{MinVertex Cover} & \quad k \\
& \quad \text{size of smallest vertex cover in } G \\
n & \quad \text{number of vertices in } G \\
k & \quad \text{size of largest independent set in } G \\
n-k & \quad \text{size of largest independent set in } G
\end{align*}
**3Color**

Input: \(G = (V, E)\)

Can we color vertices red, green, blue so that every edge touches two colors?

```
<table>
<thead>
<tr>
<th>3SAT</th>
<th>3COLOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Phi)</td>
<td>(G)</td>
</tr>
<tr>
<td>3CNF Boolean formula</td>
<td>graph</td>
</tr>
<tr>
<td>transform in (O(n)) time</td>
<td>(G) is 3-colorable</td>
</tr>
<tr>
<td>(\Phi) is satisfiable</td>
<td>(\Phi) is not satisfiable</td>
</tr>
</tbody>
</table>
```

**Truth gadget:**

**Variable gadget:**

```
SAME vertex
```

**Clause gadget:**

\((a \lor b \lor c)\)
Hamiltonian Cycle

Input: Graph $G = (V, E)$ directed

Is there a simple cycle in $G$ through every vertex

Reduce From Vertex Cover
\[ G = \begin{array}{c}
\circ \\
\text{ } \\
\bullet
\end{array} \]

\[ k = 2 \]

**Vertex Cover**

\[ G \text{ graph} \xrightarrow{k \text{ integer}} \text{ transform in } O(V+E) \text{ time} \xrightarrow{H \text{ graph}} \]

**Directed HamCycle**

\[ \begin{array}{c}
\text{TRUE} \\
H \text{ has a Ham. cycle} \\
\text{FALSE} \\
H \text{ has no Ham. cycle}
\end{array} \]

\[ \begin{array}{c}
\text{TRUE} \\
G \text{ has a vertex cover of size } k \\
\text{FALSE} \\
G \text{ has no vertex cover of size } k
\end{array} \]

**Edge Gadgets**

\[ u \xrightarrow{w} v \]

\[ (u, v, \text{out}) \]

\[ (u, v, \text{in}) \]

**Vertex Gadgets**

Connect all incident edge gadget

**Cover gadget**

\[ k \text{ vertices connected to vertex chains} \]