Let $L$ be an arbitrary regular language.

1. Prove that the language $\text{insert}_1(L) := \{xy | x \in L \}$ is regular.

Intuitively, $\text{insert}_1(L)$ is the set of all strings that can be obtained from strings in $L$ by inserting exactly one 1. For example, if $L = \{\epsilon, 00K!\}$, then $\text{insert}_1(L) = \{1, 100K!, 010K!, 001K!, 00K1!, 00K!1\}$.

2. Prove that the language $\text{delete}_1(L) := \{xy \mid x1y \in L\}$ is regular.

Intuitively, $\text{delete}_1(L)$ is the set of all strings that can be obtained from strings in $L$ by deleting exactly one 1. For example, if $L = \{101101, 00, \epsilon\}$, then $\text{delete}_1(L) = \{01101, 10101, 10110\}$.

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Work on these later: (In fact, these might be easier than problems 1 and 2.)

3. Consider the following recursively defined function on strings:

$$\text{stutter}(w) := \begin{cases} 
\epsilon & \text{if } w = \epsilon \\
aa \cdot \text{stutter}(x) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x 
\end{cases}$$

Intuitively, $\text{stutter}(w)$ doubles every symbol in $w$. For example:

- $\text{stutter(MOJO)} = \text{MMOOJJOO}$
- $\text{stutter(QUICKSILVER)} = \text{QQUICCSILVERQ}$

Let $L$ be an arbitrary regular language.

(a) Prove that the language $\text{stutter}^{-1}(L) := \{w \mid \text{stutter}(w) \in L\}$ is regular.

(b) Prove that the language $\text{stutter}(L) := \{\text{stutter}(w) \mid w \in L\}$ is regular.

4. Consider the following recursively defined function on strings:

$$\text{evens}(w) := \begin{cases} 
\epsilon & \text{if } w = \epsilon \\
\epsilon & \text{if } w = a \text{ for some symbol } a \\
b \cdot \text{evens}(x) & \text{if } w = abx \text{ for some symbols } a \text{ and } b \text{ and some string } x 
\end{cases}$$

Intuitively, $\text{evens}(w)$ skips over every other symbol in $w$. For example:

- $\text{evens(EXPELLIARMUS)} = \text{XELAMS}$
- $\text{evens(AVADA\kern.05em\scriptstyle\check{\kern-.15em\kern.05emKEDAVRA})} = \text{V\kern-.15em\scriptstyle\check{\kern-.15em\kern.05emD\kern-.15em\scriptstyle\check{\kern-.15em\kern.05emEAR}}}$

Once again, let $L$ be an arbitrary regular language.

(a) Prove that the language $\text{evens}^{-1}(L) := \{w \mid \text{evens}(w) \in L\}$ is regular.

(b) Prove that the language $\text{evens}(L) := \{\text{evens}(w) \mid w \in L\}$ is regular.