Announcement: Conflict final will be held between 1:30pm and 4:30pm on May 10, 2016. If you need to take the conflict final exam, you need to inform us by end of today.

Problem 1. [Category: Proof] A CNF formula $\varphi$ is in $k$-CNF form if each clause of $\varphi$ has exactly $k$ literals. The $k$-SAT problem is to decide if a given $k$-CNF formula is satisfiable. In this problem we will reduce 3-SAT to 4-SAT. Note that we have seen in lecture how to reduce 4-SAT (and more generally SAT) to 3-SAT.

1. Suppose $\varphi$ is a 3-CNF formula. Consider the following reduction where we add a new variable $u$ and replace each clause $c = (\ell_1 \lor \ell_2 \lor \ell_3)$ by a new clause $c' = (\ell_1 \lor \ell_2 \lor \ell_3 \lor u)$. Note that we are using the same variable for each clause. Let $\varphi'$ be the new formula obtained from $\varphi$ via this reduction. Prove that $\varphi'$ is satisfiable if $\varphi$ is satisfiable. Give an example to show that $\varphi'$ is satisfiable but $\varphi$ is not satisfiable.

2. Obtain a correct reduction by altering the preceding one and prove that $\varphi'$ is satisfiable if and only if $\varphi$ is satisfiable.

Problem 2. [Category: Proof] A path $P$ in a directed graph $G$ is called a Hamiltonian path if it contains all the vertices of $G$. The Hamiltonian Path problem is the following: given $G$, does $G$ contain a Hamiltonian path? The Longest $s$-$t$ Path problem is the following: given a directed graph $G = (V, E)$ two nodes $s, t \in V$ and an integer $k$, is there a simple path of length at least $k$ from $s$ to $t$ in $G$?

1. Assuming that you have a black box algorithm for the Longest $s$-$t$ Path problem describe a polynomial-time algorithm for the Hamiltonian Path problem.

2. Did you use a mapping reduction for the preceding part? If not, give a mapping reduction from the Hamiltonian Path problem to the Longest $s$-$t$ Path problem. That is, given $G$ your reduction should output a graph $G' = (V', E')$, two nodes $s, t \in V'$ and an integer $k$ such that $G'$ has an $s$-$t$ simple path of length at least $k$ if and only if $G$ has a Hamiltonian Path.

Problem 3. [Category: Comprehension+Proof] Self-reduction. We focus on decision problems even when the underlying problem we are interested in is an optimization problem. For most problems of interest we can in fact show that a polynomial-time algorithm for the decision problem also implies a polynomial-time algorithm for the corresponding optimization problem. To illustrate this consider the maximum independent set (MIS) problem.

1. Suppose you are given a algorithm that given a graph $H$ and integer $\ell$ outputs whether $H$ has an independent set of size at least $\ell$. Using this algorithm as a black box, describe a polynomial time algorithm that given a graph $G$ and integer $k$ outputs an independent set of size $k$ in $G$ if it has one. Note that you can use the black box algorithm more than once. {Hint: What happens if you remove a vertex $v$ and the independent set size does not decrease? What if it does?}

2. How would you efficiently find a maximum independent set in a given graph $G$ using the black box?