Deterministic Finite Automata (DFAs)

Lecture 3
January 24, 2017
Part I

DFA Introduction
DFAs also called Finite State Machines (FSMs)

- The “simplest” model for computers?
- State machines that are very common in practice.
  - Vending machines
  - Elevators
  - Digital watches
  - Simple network protocols
- Programs with fixed memory
A simple program

Program to check if a given input string $w$ has odd length

```c
int $n = 0$
While input is not finished
  read next character $c$
  $n \leftarrow n + 1$
endWhile
If ($n$ is odd) output YES
Else output NO
```
A simple program

Program to check if a given input string $w$ has odd length

```
int n = 0
While input is not finished
    read next character c
    n ← n + 1
endWhile
If (n is odd) output YES
Else output NO
```

```
bit x = 0
While input is not finished
    read next character c
    x ← flip(x)
endWhile
If (x = 1) output YES
Else output NO
```
Another view

- Machine has input written on a *read-only* tape
- Start in specified start state
- Start at left, scan symbol, change state and move right
- Circled states are *accepting*
- Machine *accepts* input string if it is in an accepting state after scanning the last symbol.
Definition 4. A deterministic finite automaton (DFA) is $M = (Q, \Sigma, \delta, s, A)$ where

- $Q$ is a finite set whose elements are called states,
- $\Sigma$ is a finite set called the input alphabet,
- $\delta: Q \times \Sigma \rightarrow Q$ is the transition function,
- $s \in Q$ is the start state,
- $A \subseteq Q$ is the set of accepting/final states.

Definition 5. For a DFA $M = (Q, \Sigma, \delta, s, A)$, string $w = w_1w_2 \ldots w_k$ where for each $i$, $w_i \in \Sigma$, and states $p, q \in Q$, we say $p \xrightarrow{w} M q$ if there is a sequence of states $r_0, r_1, \ldots, r_k$ such that (a) $r_0 = p$, (b) for each $i$, $(r_i, w_{i+1}) = r_{i+1}$, and (c) $r_k = q$.

Problem 4. Prove that for any state $p$, and string $w \in \Sigma^*$, there is a unique state $q$ such that $p \xrightarrow{w} M q$.

Notation. $\xrightarrow{w} M (p, w) = q$ where $p \xrightarrow{w} M q$.

Definition 6. Consider a DFA $M = (Q, \Sigma, \delta, s, A)$.

- $M$ accepts string $w \in \Sigma^*$ if $M$ accepts $w$.
- The language accepted/recognized by a DFA $M$ is $L(M) = \{ w \in \Sigma^* | M \text{ accepts } w \}$.
- A set $L \subseteq \Sigma^*$ is said to be accepted/recognized by $M$ if $L = L(M)$.

Problem 5.

1. Which of the following is true?
   - $B \xrightarrow{!} M B$
   - $A \xrightarrow{01} M D$
   - $D \xrightarrow{111} M C$
   - $A \xrightarrow{101} M 2B$

2. What is the following?
   - $\xrightarrow{!} M (A, 1011) = q_0$
   - $\xrightarrow{!} M (B, 010) = q_1$
   - $\xrightarrow{!} M (C, 100) = q_2$

3. What is $L(M)$?

4. What is the language recognized if we change the initial state to $B$?

5. What is the language recognized if we change the set of final states to be $\{B\}$ (with initial state $A$)?
Graphical Representation

Where does 001 lead? 10010?
Graphical Representation

![ DFA Diagram ]

- Where does 001 lead? 10010?
- Which strings end up in accepting state?

**Definition 4.** A deterministic finite automaton (DFA) is $M = (Q, \mathcal{V}, \delta, s, A)$ where

- $Q$ is a finite set whose elements are called states,
- $\mathcal{V}$ is a finite set called the input alphabet,
- $\delta: Q \times \mathcal{V} \rightarrow Q$ is the transition function,
- $s \in Q$ is the start state,
- $A \subseteq Q$ is the set of accepting/final states.

**Definition 5.** For a DFA $M = (Q, \mathcal{V}, \delta, s, A)$, string $w = w_1w_2\ldots w_k$, where for each $i$, $w_i \in \mathcal{V}$, and states $p, q \in Q$, $w$ is a walk from $p$ to $q$ if there is a sequence of states $r_0, r_1, \ldots, r_k$ such that

- $r_0 = p$,
- for each $i$, $(r_i, w_{i+1}) = r_{i+1}$,
- $r_k = q$.

**Problem 4.** Prove that for any state $p$, and string $w \in \mathcal{V}^*$, there is a unique state $q$ such that $p \xrightarrow{w} M q$.

**Notation.** $\xrightarrow{w} M q$ where $p \xrightarrow{w} M q$.

**Definition 6.** Consider a DFA $M = (Q, \mathcal{V}, \delta, s, A)$.

- $M$ accepts string $w \in \mathcal{V}^*$ if $M$ accepts $w$.
- The language accepted/recognized by a DFA $M$ is $L(M) = \{ w \in \mathcal{V}^* | M \text{ accepts } w \}$.

**Problem 5.**

1. Which of the following is true?
   - $B \not\xrightarrow{!} M B$
   - $A \not\xrightarrow{01} M D$
   - $D \not\xrightarrow{111} M C$
   - $A \not\xrightarrow{101} M 2B$

2. What is the following?
   - $\xrightarrow{2} M (A, 1011) = q_0$
   - $\xrightarrow{2} M (B, 010) = q_1$
   - $\xrightarrow{2} M (C, 100) = q_2$
   - $\xrightarrow{2} M (D, 001) = q_3$

3. What is $L(M)$?

4. What is the language recognized if we change the initial state to $B$?

5. What is the language recognized if we change the set of final states to be $\{B\}$ (with initial state $A$)?
Definition 4. A deterministic finite automaton (DFA) is $M = (Q, \mathcal{A}, s, A)$ where
- $Q$ is a finite set whose elements are called states,
- $\mathcal{A}$ is a finite set called the input alphabet,
- $\delta: Q \times \mathcal{A} \rightarrow Q$ is the transition function,
- $s \in Q$ is the start state,
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Definition 5. For a DFA $M = (Q, \mathcal{A}, s, A)$, string $w = w_1w_2 \cdots w_k$, where for each $i$, $w_i \in \mathcal{A}$, and states $p, q \in Q$, we say $p \xrightarrow{w} M q$ if there is a sequence of states $r_0, r_1, \ldots, r_k$ such that (a) $r_0 = p$, (b) for each $i$, $(r_i, w_{i+1}) = r_{i+1}$, and (c) $r_k = q$.

Problem 4. Prove that for any state $p$, and string $w \in \mathcal{A}^*$, there is a unique state $q$ such that $p \xrightarrow{w} M q$.

Notation. $\xrightarrow{M} (p, w) = q$ where $p \xrightarrow{w} M q$.

Definition 6. Consider a DFA $M = (Q, \mathcal{A}, s, A)$.
- $M$ accepts string $w \in \mathcal{A}^*$ if $M$ accepts $w$.
- The language accepted/recognized by a DFA $M$ is $L(M) = \{ w \in \mathcal{A}^* \mid M$ accepts $w \}$.
- A set $L \subseteq \mathcal{A}^*$ is said to be accepted/recognized by $M$ if $L = L(M)$.

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   - $B \not\xrightarrow{M} B$
   - $A \xrightarrow{01} M D$
   - $D \xrightarrow{111} M C$
   - $A \xrightarrow{101} M 2 B$

   2. What is the following?
   - $\xrightarrow{M} (A, 1011) = q_3$
   - $\xrightarrow{M} (B, 010) = q_3$
   - $\xrightarrow{M} (C, 100) = q_0$

   Figure 1: DFA $M$ for problem 5

3. What is $L(M)$?
4. What is the language recognized if we change the initial state to $B$?
5. What is the language recognized if we change the set of final states to be $\{B\}$ (with initial state $A$)?

- Where does 001 lead? 10010?
- Which strings end up in accepting state?
- Can you prove it?
Graphical Representation

Where does 001 lead? 10010?

Which strings end up in accepting state?

Can you prove it?

Every string \( w \) has a unique walk that it follows from a given state \( q \) by reading one letter of \( w \) from left to right.
A deterministic finite automaton (DFA) is a 4-tuple $M = (Q, \mathcal{A}, s, A)$ where

- $Q$ is a finite set whose elements are called states,
- $\mathcal{A}$ is a finite set called the input alphabet,
- $\delta: Q \times \mathcal{A} \rightarrow Q$ is the transition function,
- $s \in Q$ is the start state,
- $A \subset Q$ is the set of accepting/final states.

Definition 5. For a DFA $M = (Q, \mathcal{A}, s, A)$, string $w = w_1w_2\ldots w_k$, where for each $i$, $w_i \in \mathcal{A}$, and states $p, q \in Q$, we say $p \overset{w}{\Rightarrow} M q$ if there is a sequence of states $r_0, r_1, \ldots, r_k$ such that

- $r_0 = p$,
- for each $i$, $(r_i, w_{i+1}) = r_{i+1}$,
- $r_k = q$.

Problem 4. Prove that for any state $p$, and string $w \in \mathcal{A}^*$, there is a unique state $q$ such that $p \overset{w}{\Rightarrow} M q$.

Notation. $\overset{w}{\Rightarrow} M q$ where $p \overset{w}{\Rightarrow} M q$.

Definition 6. Consider a DFA $M = (Q, \mathcal{A}, s, A)$.

- $M$ accepts string $w \in \mathcal{A}^*$ iff the unique walk starting at the start state and spelling out $w$ ends in an accepting state.

- The language accepted/recognized by a DFA $M$ is $L(M) = \{w \in \mathcal{A}^* | M \text{ accepts } w\}$.

- A set $L \subset \mathcal{A}^*$ is said to be accepted/recognized by $M$ if $L = L(M)$.
Definition 4. A deterministic finite automaton (DFA) is $M = (Q, \Sigma, \delta, s, A)$ where
- $Q$ is a finite set whose elements are called states,
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- (a) $r_0 = p$, (b) for each $i$, $(r_i, w_{i+1}) = r_{i+1}$, and (c) $r_k = q$.

Problem 4. Prove that for any state $p$, and string $w \in \Sigma^*$, there is a unique state $q$ such that $p \xrightarrow{w} M q$.

Notation. $\xrightarrow{w} M q$ where $p \xrightarrow{w} M q$.

Definition 6. Consider a DFA $M = (Q, \Sigma, \delta, s, A)$.
- $M$ accepts string $w \in \Sigma^*$ if $\xrightarrow{w} M s$ and $s \in A$.
- The language accepted/recognized by a DFA $M$ is $L(M) = \{ w \mid M$ accepts $w \}$.

Problem 5. 1. Which of the following is true?
   - $B \xrightarrow{!} M B$
   - $A \xrightarrow{01} M D$
   - $D \xrightarrow{111} M C$
   - $A \xrightarrow{101} M 2$

   2. What is the following?
   - $\xrightarrow{2} M 2(A, 1011) = q_0$
   - $\xrightarrow{2} M 2(B, 010) = q_1$
   - $\xrightarrow{2} M 2(C, 100) = q_2$

   Figure 1: DFA $M$ for problem 5

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Warning

“$M$ accepts language $L$” does not mean simply that that $M$ accepts each string in $L$.

It means that $M$ accepts each string in $L$ and no others. Equivalently $M$ accepts each string in $L$ and does not accept/rejects strings in $\Sigma^* \setminus L$.
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$M$ “recognizes” $L$ is a better term but “accepts” is widely accepted (and recognized) (joke attributed to Lenny Pitt)
A deterministic finite automata (DFA) $M = (Q, \Sigma, \delta, s, A)$ is a five tuple where

- $Q$ is a finite set whose elements are called states,
- $\Sigma$ is a finite set called the input alphabet,
- $\delta: Q \times \Sigma \rightarrow Q$ is the transition function,
- $s \in Q$ is the start state,
- $A \subseteq Q$ is the set of accepting/final states.

Common alternate notation:
- $q_0$ for start state,
- $F$ for final states.
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Common alternate notation: $q_0$ for start state, $F$ for final states.
Example

A deterministic finite automaton (DFA) is $M = (Q, \Sigma, , s, A)$ where
- $Q$ is a finite set whose elements are called states,
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- $(a) r_0 = p$,
- $(b) \forall i, (r_i, w_{i+1}) = r_{i+1}$,
- $(c) r_k = q$.

Problem 4. Prove that for any state $p$, and string $w \in \Sigma^*$, there is a unique state $q$ such that $p w \rightarrow M q$.

Notation. $\rightarrow M (p, w) = q$ where $p w \rightarrow M q$.

Definition 6. Consider a DFA $M = (Q, \Sigma, , s, A)$.
- $M$ accepts string $w \in \Sigma^*$ if $M$ accepts $\epsilon \cdot w$.
- The language accepted/recognized by a DFA $M$ is $L(M) = \{ w \in \Sigma^* \mid M$ accepts $w \}$. A set $L \subseteq \Sigma^*$ is said to be accepted/recognized by $M$ if $L = L(M)$.
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Problem 4. Prove that for any state \( p \), and string \( w \in \Sigma^* \), there is a unique state \( q \) such that \( p \xrightarrow{w} M q \).

Notation. \( \xrightarrow{w} M \) \( (p, w) \) = \( q \) where \( p \xrightarrow{w} M q \).

Definition 6. Consider a DFA \( M = (Q, \Sigma, \delta, s, A) \).

- \( M \) accepts string \( w \in \Sigma^* \) if \( M \) accepts \( w \).
- The language accepted/recognized by a DFA \( M \) is \( L(M) = \{ w \in \Sigma^* \mid M \) accepts \( w \} \).
- A set \( L \subseteq \Sigma^* \) is said to be accepted/recognized by \( M \) if \( L = L(M) \).

Problem 5.
1. Which of the following is true?
   - \( B \xrightarrow{} M B \)
   - \( A \xrightarrow{01} M D \)
   - \( D \xrightarrow{111} M C \)
   - \( A \xrightarrow{101} M 2 B \)

2. What is the following?
   - \( \xrightarrow{} M 2 (A, 1011) = \)
   - \( \xrightarrow{} M 2 (B, 010) = \)
   - \( \xrightarrow{} M 2 (C, 100) = \)

3. What is \( L(M) \)?

4. What is the language recognized if we change the initial state to \( B \)?

5. What is the language recognized if we change the set of final states to be \( \{B\} \) (with initial state \( A \))?
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Definition 6. Consider a DFA $M = (Q, \Sigma, \delta, s, A)$.
- $M$ accepts string $w \in \Sigma^*$ if $w \xrightarrow{} M s$.
- The language accepted/recognized by a DFA $M$ is $L(M) = \{ w \in \Sigma^* | w \xrightarrow{} M q \}$.

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![DFA Diagram]

- $Q = \{q_0, q_1, q_1, q_3\}$
- $\Sigma = \{0, 1\}$
**Definition 4.** A deterministic finite automaton (DFA) is $M = (Q, \Sigma, \delta, s, A)$ where
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1. $r_0 = p$,
2. for each $i$, $(r_i, w_{i+1}) = r_{i+1}$,
3. $r_k = q$.

**Problem 4.** Prove that for any state $p$, and string $w \in \Sigma^*$, there is a unique state $q$ such that $p \xrightarrow{w} M q$.

**Notation.** $\xrightarrow{w} M q$ where $p \xrightarrow{w} M q$.

**Definition 6.** Consider a DFA $M = (Q, \Sigma, \delta, s, A)$.
- $M$ accepts string $w \in \Sigma^*$ if $M(w) \in A$.
- The language accepted/recognized by a DFA $M$ is $L(M) = \{w \in \Sigma^* | M\text{ accepts } w\}$.
- A set $L \subseteq \Sigma^*$ is said to be accepted/recognized by $M$ if $L = L(M)$.

**Problem 5.**
1. Which of the following is true?
   - $B \not\xrightarrow{!} M B$
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   - $A \xrightarrow{101} M 2B$

2. What is the following?
   - $\xrightarrow{A,B} M (A,1011) =$
   - $\xrightarrow{B,C} M (B,010) =$
   - $\xrightarrow{C,D} M (C,100) =$

3. What is $L(M)$?

4. What is the language recognized if we change the initial state to $B$?

5. What is the language recognized if we change the set of final states to be $\{B\}$ (with initial state $A$)?
A deterministic finite automaton (DFA) is  

\[ M = (Q, \Sigma, \delta, s, A) \]

where

- \( Q \) is a finite set whose elements are called states,
- \( \Sigma \) is a finite set called the input alphabet,
- \( \delta : Q \times \Sigma \rightarrow Q \) is the transition function,
- \( s \in Q \) is the start state,
- \( A \subseteq Q \) is the set of accepting/final states.

**Problem 4.** Prove that for any state \( p \), and string \( w \in \Sigma^* \), there is a unique state \( q \) such that \( p \xrightarrow{w} M q \).

**Notation.** \( \xrightarrow{w} M \) \((p, w) = q \) where \( p \xrightarrow{w} M q \).

**Definition 5.** For a DFA \( M = (Q, \Sigma, \delta, s, A) \), string \( w = w_1w_2\ldots w_k \), where for each \( i \), \( w_i \in \Sigma \), and states \( p, q \in Q \), \( p \xrightarrow{w_i} M q \) if there is a sequence of states \( r_0, r_1, \ldots, r_k \) such that

1. \( r_0 = p \),
2. for each \( i \), \( (r_i, w_{i+1}) = r_{i+1} \), and
3. \( r_k = q \).

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- $s \in Q$ is the start state,
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Definition 5. For a DFA $M = (Q, \Sigma, \delta, s, A)$, string $w = w_1 w_2 \cdots w_k$, where for each $i$, $w_i \in \Sigma$, and states $p, q \in Q$, we say $p \xrightarrow{w} M q$ if there is a sequence of states $r_0, r_1, \ldots, r_k$ such that (a) $r_0 = p$, (b) for each $i$, $(r_i, w_{i+1}) = r_{i+1}$, and (c) $r_k = q$.

Problem 4. Prove that for any state $p$, and string $w \in \Sigma^*$, there is a unique state $q$ such that $p \xrightarrow{w} M q$.

Notation. $\xrightarrow{\cdot} M (p, w) = q$ where $p \xrightarrow{w} M q$.

Definition 6. Consider a DFA $M = (Q, \Sigma, \delta, s, A)$.

- $M$ accepts string $w \in \Sigma^*$ if $\xrightarrow{\cdot} M (s, w) \in A$.
- The language accepted/recognized by a DFA $M$ is $L(M) = \{ w \in \Sigma^* | \xrightarrow{\cdot} M (s, w) \in A \}$.
- A set $L \subseteq \Sigma^*$ is said to be accepted/recognized by $M$ if $L = L(M)$.

Problem 5.
1. Which of the following is true?
   - $B \xrightarrow{!} M B$
   - $A \xrightarrow{01} M D$
   - $D \xrightarrow{111} M C$
   - $A \xrightarrow{101} M 2B$

2. What is the following?
   - $\xrightarrow{\cdot} M 2(A, 1011) =$
   - $\xrightarrow{\cdot} M 2(B, 010) =$
   - $\xrightarrow{\cdot} M 2(C, 100) =$

3. What is $L(M)$?
4. What is the language recognized if we change the initial state to $B$?
5. What is the language recognized if we change the set of final states to be $\{B\}$ (with initial state $A$)?

![Diagram of DFA](image-url)
Example

- \( Q = \{q_0, q_1, q_1, q_3\} \)
- \( \Sigma = \{0, 1\} \)
- \( \delta \)
- \( s = q_0 \)
- \( A = \)
Definition 4. A deterministic finite automaton (DFA) is $M = (Q, \Sigma, \delta, s, A)$ where
- $Q$ is a finite set whose elements are called states,
- $\Sigma$ is a finite set called the input alphabet,
- $\delta : Q \times \Sigma \rightarrow Q$ is the transition function,
- $s \in Q$ is the start state,
- $A \subset Q$ is the set of accepting/final states.

Definition 5. For a DFA $M = (Q, \Sigma, \delta, s, A)$, string $w = w_1w_2 \cdots w_k$, where for each $i$, $w_i \in \Sigma$, and states $p, q \in Q$, we say $p \xrightarrow{w} M q$ if there is a sequence of states $r_0, r_1, \ldots, r_k$ such that
1. $r_0 = p$,
2. for each $i$, $(r_i, w_{i+1}) = r_{i+1}$,
3. $r_k = q$.

Problem 4. Prove that for any state $p$, and string $w \in \Sigma^*$, there is a unique state $q$ such that $p \xrightarrow{w} M q$.

Notation. $\xrightarrow{M} (p, w) = q$ where $p \xrightarrow{w} M q$.

Definition 6. Consider a DFA $M = (Q, \Sigma, \delta, s, A)$.
- $M$ accepts string $w \in \Sigma^*$ if $M$ accepts $w$.
- The language accepted/recognized by a DFA $M$ is $L(M) = \{ w \in \Sigma^* | M$ accepts $w \}$.
- A set $L \subset \Sigma^*$ is said to be accepted/recognized by $M$ if $L = L(M)$.

Problem 5.
1. Which of the following is true?
   - $B \xrightarrow{\epsilon} M B$
   - $A \xrightarrow{101} M D$
   - $D \xrightarrow{111} M C$
   - $A \xrightarrow{101} M 2B$
2. What is the following?
   - $\xrightarrow{M} (A, 101) = q_0$
   - $\xrightarrow{M} (B, 010) = q_0$
   - $\xrightarrow{M} (C, 100) = q_0$

Figure 1: DFA $M$ for problem 5

3. What is $L(M)$?
4. What is the language recognized if we change the initial state to $B$?
5. What is the language recognized if we change the set of final states to be $\{B\}$ (with initial state $A$)?

Example

- $Q = \{q_0, q_1, q_1, q_3\}$
- $\Sigma = \{0, 1\}$
- $\delta$
- $s = q_0$
- $A = \{q_0\}$
Extending the transition function to strings

Given DFA $M = (Q, \Sigma, \delta, s, A)$, $\delta(q, a)$ is the state that $M$ goes to from $q$ on reading letter $a$.

Useful to have notation to specify the unique state that $M$ will reach from $q$ on reading string $w$. 
Extending the transition function to strings

Given DFA \( M = (Q, \Sigma, \delta, s, A) \), \( \delta(q, a) \) is the state that \( M \) goes to from \( q \) on reading letter \( a \).

Useful to have notation to specify the unique state that \( M \) will reach from \( q \) on reading string \( w \).

Transition function \( \delta^* : Q \times \Sigma^* \rightarrow Q \) defined inductively as follows:

- \( \delta^*(q, \epsilon) = q \) if \( w = \epsilon \)
- \( \delta^*(q, ax) = \delta^*(\delta(q, a), x) \) if \( w = ax \).
Definition

The language \( L(M) \) accepted by a DFA \( M = (Q, \Sigma, \delta, s, A) \) is

\[ \{ w \in \Sigma^* \mid \delta^*(s, w) \in A \}. \]
Definition 4. A deterministic finite automaton (DFA) is \( M = (Q, \mathcal{A}, \delta, s, A) \) where

- \( Q \) is a finite set whose elements are called states,
- \( \mathcal{A} \) is a finite set called the input alphabet,
- \( \delta : Q \times \mathcal{A} \rightarrow Q \) is the transition function,
- \( s \in Q \) is the start state,
- \( A \subseteq Q \) is the set of accepting/final states.

Definition 5. For a DFA \( M = (Q, \mathcal{A}, \delta, s, A) \), string \( w = w_1w_2 \cdots w_k \), where for each \( i \), \( w_i \in \mathcal{A} \), and states \( p, q \in Q \), \( w \in M \) if there is a sequence of states \( r_0, r_1, \ldots, r_k \) such that:

1. \( r_0 = p \),
2. for each \( i \), \( (r_i, w_{i+1}) = r_{i+1} \), and
3. \( r_k = q \).

Problem 4. Prove that for any state \( p \) and string \( w \in \mathcal{A} \), there is a unique state \( q \) such that \( p \xrightarrow{w} M q \).

Notation. \( \xrightarrow{\epsilon} M (p, w) = q \) where \( p \xrightarrow{w} M q \).

Definition 6. Consider a DFA \( M = (Q, \mathcal{A}, \delta, s, A) \).

- \( M \) accepts string \( w \in \mathcal{A} \) if \( M \) accepts \( w \in \mathcal{A} \).
- The language accepted/recognized by a DFA \( M \) is \( L(M) = \{ w \in \mathcal{A} \mid M \) accepts \( w \} \).
- A set \( L \subseteq \mathcal{A} \) is said to accepted/recognized by \( M \) if \( L = L(M) \).

Problem 5. 1. Which of the following is true?
   - \( B \xrightarrow{} M B \)
   - \( A \xrightarrow{01} M D \)
   - \( D \xrightarrow{111} M C \)
   - \( A \xrightarrow{101} M 2B \)

2. What is the following?
   - \( \xrightarrow{} M 2(A, 1011) = q_1 \)
   - \( \xrightarrow{} M 2(B, 010) = q_2 \)
   - \( \xrightarrow{} M 2(C, 100) = q_3 \)

3. What is \( L(M) \)?

4. What is the language recognized if we change the initial state to \( B \)?

5. What is the language recognized if we change the set of final states to be \( \{B\} \) (with initial state \( A \))? 

What is:

- \( \delta^*(q_1, \epsilon) = q_1 \)
- \( \delta^*(q_0, 1011) = q_2 \)
- \( \delta^*(q_1, 010) \)
- \( \delta^*(q_4, 10) \)
Example continued

Definition 4. A deterministic finite automaton (DFA) is
\[ M = (Q, \mathcal{A}, s, A) \]
where
- \( Q \) is a finite set whose elements are called states,
- \( \mathcal{A} \) is a finite set called the input alphabet,
- \( : Q \times \mathcal{A} \rightarrow Q \) is the transition function,
- \( s \in Q \) is the start state,
- \( A \subseteq Q \) is the set of accepting/final states.

Definition 5. For a DFA \( M = (Q, \mathcal{A}, s, A) \), string \( w = w_1 w_2 \cdots w_k \), where for each \( i \), \( w_i \in \mathcal{A} \), and states \( p, q \in Q \), \( p \xrightarrow{w} M q \) if there is a sequence of states \( r_0, r_1, \ldots, r_k \) such that:
1. \( r_0 = p \),
2. for each \( i \), \( (r_i, w_{i+1}) = r_{i+1} \), and
3. \( r_k = q \).

Problem 4. Prove that for any state \( p \) and string \( w \in \mathcal{A}^* \), there is a unique state \( q \) such that \( p \xrightarrow{w} M q \).

Notation. \( \xrightarrow{w} M (p, w) = q \) where \( p \xrightarrow{w} M q \).

Problem 5.
1. Which of the following is true?
   - \( B \xrightarrow{!} M B \)
   - \( A \xrightarrow{01} M D \)
   - \( D \xrightarrow{111} M C \)
   - \( A \xrightarrow{101} M 2B \)

2. What is the following?
   - \( \xrightarrow{w} M (A, 1011) = \)
   - \( \xrightarrow{w} M (B, 010) = \)
   - \( \xrightarrow{w} M (C, 100) = \)

   ![DFA Diagram]

   - What is \( L(M) \) if start state is changed to \( q_1 \)?
   - What is \( L(M) \) if final/accept states are set to \( \{q_2, q_3\} \) instead of \( \{q_0\} \)?
Advantages of formal specification

- Necessary for proofs
- Necessary to specify abstractly for class of languages

**Exercise:** Prove by induction that for any two strings $u, v$, any state $q$, $\delta^*(q, uv) = \delta^*(\delta^*(q, u), v)$. 
How do we design a DFA $M$ for a given language $L$? That is $L(M) = L$.

- DFA is a like a program that has fixed amount of memory independent of input size.
- The memory of a DFA is encoded in its states.
- The state/memory must capture enough information from the input seen so far that it is sufficient for the suffix that is yet to be seen (note that DFA cannot go back).
DFA Construction: Example

Assume $\Sigma = \{0, 1\}$

- $L = \emptyset$, $L = \Sigma^*$, $L = \{\epsilon\}$, $L = \{0\}$.
DFA Construction: Example

Assume $\Sigma = \{0, 1\}$

- $L = \emptyset$, $L = \Sigma^*$, $L = \{\epsilon\}$, $L = \{0\}$.
- $L = \{w \in \{0, 1\}^* \mid |w| \text{ is divisible by } 5\}$
DFA Construction: Example

Assume $\Sigma = \{0, 1\}$

- $L = \emptyset$, $L = \Sigma^*$, $L = \{\epsilon\}$, $L = \{0\}$.
- $L = \{w \in \{0, 1\}^* \mid |w| \text{ is divisible by 5}\}$
- $L = \{w \in \{0, 1\}^* \mid w \text{ ends with 01}\}$

The DFA construction for these languages is illustrated in the diagram with the input string 101001.
Assume $\Sigma = \{0, 1\}$

- $L = \emptyset$, $L = \Sigma^*$, $L = \{\varepsilon\}$, $L = \{0\}$.
- $L = \{w \in \{0, 1\}^* \mid |w| \text{ is divisible by } 5\}$
- $L = \{w \in \{0, 1\}^* \mid w \text{ ends with } 01\}$
- $L = \{w \in \{0, 1\}^* \mid w \text{ contains } 001 \text{ as substring}\}$
Assume $\Sigma = \{0, 1\}$

- $L = \emptyset$, $L = \Sigma^*$, $L = \{\epsilon\}$, $L = \{0\}$.
- $L = \{w \in \{0, 1\}^* \mid |w| \text{ is divisible by 5}\}$
- $L = \{w \in \{0, 1\}^* \mid w \text{ ends with 01}\}$
- $L = \{w \in \{0, 1\}^* \mid w \text{ contains 001 as substring}\}$
- $L = \{w \in \{0, 1\}^* \mid w \text{ contains 001 or 010 as substring}\}$
DFA Construction: Example

Assume $\Sigma = \{0, 1\}$

- $L = \emptyset$, $L = \Sigma^*$, $L = \{\epsilon\}$, $L = \{0\}$.
- $L = \{w \in \{0, 1\}^* \mid |w| \text{ is divisible by } 5\}$
- $L = \{w \in \{0, 1\}^* \mid w \text{ ends with } 01\}$
- $L = \{w \in \{0, 1\}^* \mid w \text{ contains } 001 \text{ as substring}\}$
- $L = \{w \in \{0, 1\}^* \mid w \text{ contains } 001 \text{ or } 010 \text{ as substring}\}$
- $L = \{w \mid w \text{ has a } 1 \text{ \(k\) positions from the end}\}$
 DFA Construction: Example

\[ L = \{ \text{Binary numbers congruent to 0 mod 5} \} \]
Example: \(1101011 = 107 = 2 \mod 5, \ 1010 = 10 = 0 \mod 5\)
\[ L = \{ \text{Binary numbers congruent to } 0 \mod 5 \} \]

Example: \( 1101011 = 107 = 2 \mod 5 \), \( 1010 = 10 = 0 \mod 5 \)

**Key observation:**

\( w_0 \mod 5 = a \) implies

\[ w_0 \mod 5 = 2a \mod 5 \quad \text{and} \quad w_1 \mod 5 = (2a + 1) \mod 5 \]
Part III

Product Construction and Closure Properties
Part IV

Complement
Complement

Question: If $M$ is a DFA, is there a DFA $M'$ such that $L(M') = \Sigma^* \setminus L(M)$? That is, are languages recognized by DFAs closed under complement?

![Diagram of DFA]

**Definition 4.** A deterministic finite automaton (DFA) is $M = (Q, \Sigma, \delta, s, A)$ where
- $Q$ is a finite set whose elements are called states,
- $\Sigma$ is a finite set called the input alphabet,
- $\delta: Q \times \Sigma \to Q$ is the transition function,
- $s \in Q$ is the start state,
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**Definition 5.** For a DFA $M = (Q, \Sigma, \delta, s, A)$, string $w = w_1w_2\cdots w_k$, where for each $i$, $w_i \in \Sigma$, and states $p, q \in Q$, we say $p \xrightarrow{w} M q$ if there is a sequence of states $r_0, r_1, \ldots, r_k$ such that
- $(a)$ $r_0 = p$,
- $(b)$ for each $i$, $(r_i, w_{i+1}) = r_{i+1}$, and
- $(c)$ $r_k = q$.

**Problem 4.** Prove that for any state $p$, and string $w \in \Sigma^*$, there is a unique state $q$ such that $p \xrightarrow{w} M q$.

**Notation.** $\xrightarrow{\Sigma} M (p, w) = q$ where $p \xrightarrow{w} M q$.

**Definition 6.** Consider a DFA $M = (Q, \Sigma, \delta, s, A)$.
- $M$ accepts string $w \in \Sigma^*$ if $M(s, w) \in A$.
- The language accepted/recognized by a DFA $M$ is $L(M) = \{ w \in \Sigma^* \mid M \text{ accepts } w \}$.
- A set $L \subseteq \Sigma^*$ is said to be accepted/recognized by $M$ if $L = L(M)$.
Theorem

Languages accepted by DFAs are closed under complement.

Proof.

Let $M = (Q, \Sigma, \delta, s, A)$ such that $L = L(M)$.

Let $M' = (Q, \Sigma, \delta, s, Q \setminus A)$. Claim: $L(M') = \overline{L}$.

Why?

$\delta^* M = \delta^* M'$. Thus, for every string $w$, $\delta^* M(s, w) = \delta^* M'(s, w)$.

$\delta^* M(s, w) \in A \Rightarrow \delta^* M'(s, w) \notin Q \setminus A$.

$\delta^* M(s, w) \notin A \Rightarrow \delta^* M'(s, w) \in Q \setminus A$. 
Complement

**Theorem**

Languages accepted by DFAs are closed under complement.

**Proof.**

Let $M = (Q, \Sigma, \delta, s, A)$ such that $L = L(M)$.
Let $M' = (Q, \Sigma, \delta, s, Q \setminus A)$. Claim: $L(M') = \overline{L}$. Why?
Theorem

Languages accepted by DFAs are closed under complement.

Proof.

Let $M = (Q, \Sigma, \delta, s, A)$ such that $L = L(M)$.
Let $M' = (Q, \Sigma, \delta, s, Q \setminus A)$. Claim: $L(M') = \bar{L}$. Why?
$
\delta^*_M = \delta^*_{M'}$. Thus, for every string $w$, $\delta^*_M(s, w) = \delta^*_{M'}(s, w)$.

$\delta^*_M(s, w) \in A \Rightarrow \delta^*_{M'}(s, w) \notin Q \setminus A.$
$\delta^*_M(s, w) \notin A \Rightarrow \delta^*_{M'}(s, w) \in Q \setminus A.$
Part V

Product Construction
Union and Intersection

**Question:** Are languages accepted by DFAs closed under union? That is, given DFAs $M_1$ and $M_2$ is there a DFA that accepts $L(M_1) \cup L(M_2)$? How about intersection $L(M_1) \cap L(M_2)$?
Question: Are languages accepted by DFAs closed under union? That is, given DFAs $M_1$ and $M_2$ is there a DFA that accepts $L(M_1) \cup L(M_2)$? How about intersection $L(M_1) \cap L(M_2)$?

Idea from programming: on input string $w$

- Simulate $M_1$ on $w$
- Simulate $M_2$ on $w$
- If both accept than $w \in L(M_1) \cap L(M_2)$. If at least one accepts then $w \in L(M_1) \cup L(M_2)$.
**Question:** Are languages accepted by DFAs closed under union? That is, given DFAs $M_1$ and $M_2$ is there a DFA that accepts $L(M_1) \cup L(M_2)$? How about intersection $L(M_1) \cap L(M_2)$?

Idea from programming: on input string $w$

- Simulate $M_1$ on $w$
- Simulate $M_2$ on $w$
- If both accept then $w \in L(M_1) \cap L(M_2)$. If at least one accepts then $w \in L(M_1) \cup L(M_2)$.
- **Catch:** We want a single DFA $M$ that can only read $w$ once.
Union and Intersection

**Question:** Are languages accepted by DFAs closed under union? That is, given DFAs $M_1$ and $M_2$ is there a DFA that accepts $L(M_1) \cup L(M_2)$? How about intersection $L(M_1) \cap L(M_2)$?

Idea from programming: on input string $w$

- Simulate $M_1$ on $w$
- Simulate $M_2$ on $w$
- If both accept than $w \in L(M_1) \cap L(M_2)$. If at least one accepts then $w \in L(M_1) \cup L(M_2)$.

**Catch:** We want a single DFA $M$ that can only read $w$ once.

**Solution:** Simulate $M_1$ and $M_2$ in parallel by keeping track of states of both machines.
Example

$M_2$ accepts $\#1 = \text{odd}$

$M_1$ accepts $\#0 = \text{odd}$
Example

$M_1$ accepts #0 = odd

$M_2$ accepts #1 = odd

Cross-product machine
Product construction for intersection

$M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, s_2, A_2)$

Create $M = (Q, \Sigma, \delta, s, A)$ where
Product construction for intersection

\( M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1) \) and \( M_2 = (Q_1, \Sigma, \delta_2, s_2, A_2) \)

Create \( M = (Q, \Sigma, \delta, s, A) \) where

- \( Q = \)
Product construction for intersection

\[ M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1) \text{ and } M_2 = (Q_1, \Sigma, \delta_2, s_2, A_2) \]

Create \( M = (Q, \Sigma, \delta, s, A) \) where

\[ Q = Q_1 \times Q_2 = \{(q_1, q_2) \mid q_1 \in Q_1, q_2 \in Q_2\} \]
Product construction for intersection

\[ M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1) \text{ and } M_2 = (Q_1, \Sigma, \delta_2, s_2, A_2) \]

Create \( M = (Q, \Sigma, \delta, s, A) \) where

- \( Q = Q_1 \times Q_2 = \{(q_1, q_2) \mid q_1 \in Q_1, q_2 \in Q_2\} \)
- \( s = \)
Product construction for intersection

$M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1)$ and $M_2 = (Q_1, \Sigma, \delta_2, s_2, A_2)$

Create $M = (Q, \Sigma, \delta, s, A)$ where

- $Q = Q_1 \times Q_2 = \{(q_1, q_2) | q_1 \in Q_1, q_2 \in Q_2\}$
- $s = (s_1, s_2)$
Product construction for intersection

$M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1)$ and $M_2 = (Q_1, \Sigma, \delta_2, s_2, A_2)$

Create $M = (Q, \Sigma, \delta, s, A)$ where

- $Q = Q_1 \times Q_2 = \{(q_1, q_2) \mid q_1 \in Q_1, q_2 \in Q_2\}$
- $s = (s_1, s_2)$
- $\delta : Q \times \Sigma \to Q$ where

$$\delta((q_1, q_2), a) =$$
Product construction for intersection

$M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1)$ and $M_2 = (Q_1, \Sigma, \delta_2, s_2, A_2)$

Create $M = (Q, \Sigma, \delta, s, A)$ where

- $Q = Q_1 \times Q_2 = \{(q_1, q_2) \mid q_1 \in Q_1, q_2 \in Q_2\}$
- $s = (s_1, s_2)$
- $\delta : Q \times \Sigma \to Q$ where

$$\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$$
Product construction for intersection

\[ M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1) \] and \[ M_2 = (Q_1, \Sigma, \delta_2, s_2, A_2) \]

Create \[ M = (Q, \Sigma, \delta, s, A) \] where

- \[ Q = Q_1 \times Q_2 = \{(q_1, q_2) \mid q_1 \in Q_1, q_2 \in Q_2\} \]
- \[ s = (s_1, s_2) \]
- \[ \delta : Q \times \Sigma \rightarrow Q \] where

\[ \delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a)) \]

- \[ A = \]
Product construction for intersection

\[ M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1) \text{ and } M_2 = (Q_1, \Sigma, \delta_2, s_2, A_2) \]

Create \( M = (Q, \Sigma, \delta, s, A) \) where

- \( Q = Q_1 \times Q_2 = \{(q_1, q_2) \mid q_1 \in Q_1, q_2 \in Q_2\} \)
- \( s = (s_1, s_2) \)
- \( \delta : Q \times \Sigma \rightarrow Q \) where
  \[
  \delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))
  \]

- \( A = A_1 \times A_2 = \{(q_1, q_2) \mid q_1 \in A_1, q_2 \in A_2\} \)

Theorem \( L(M) = L(M_1) \cap L(M_2) \).
Product construction for intersection

$M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1)$ and $M_2 = (Q_1, \Sigma, \delta_2, s_2, A_2)$

Create $M = (Q, \Sigma, \delta, s, A)$ where

- $Q = Q_1 \times Q_2 = \{(q_1, q_2) \mid q_1 \in Q_1, q_2 \in Q_2\}$
- $s = (s_1, s_2)$
- $\delta : Q \times \Sigma \rightarrow Q$ where
  \[
  \delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))
  \]
- $A = A_1 \times A_2 = \{(q_1, q_2) \mid q_1 \in A_1, q_2 \in A_2\}$

Theorem

$L(M) = L(M_1) \cap L(M_2)$. 
Correctness of construction

Lemma

For each string \( w \), \( \delta^*(\epsilon, w) = (\delta_1^*(q_1, w), \delta_2^*(q_2, w)) \).

Exercise: Assuming lemma prove the theorem in previous slide.

Proof of lemma by induction on \( |w| \).
Correctness of construction

**Lemma**

For each string $w$, $\delta^*(s, w) = (\delta_1^*(s_1, w), \delta_2^*(s_2, w))$.

**Exercise:** Assuming lemma prove the theorem in previous slide.
Correctness of construction

**Lemma**

For each string $w$, $\delta^*(s, w) = (\delta_1^*(s_1, w), \delta_2^*(s_2, w))$.

**Exercise:** Assuming lemma prove the theorem in previous slide. Proof of lemma by induction on $|w|$. 
Product construction for union

\( M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1) \) and \( M_2 = (Q_1, \Sigma, \delta_2, s_2, A_2) \)

Create \( M = (Q, \Sigma, \delta, s, A) \) where

- \( Q = Q_1 \times Q_2 = \{(q_1, q_2) \mid q_1 \in Q_1, q_2 \in Q_2\} \)
- \( s = (s_1, s_2) \)
- \( \delta : Q \times \Sigma \rightarrow Q \) where

\[ \delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a)) \]

- \( A = \)
Product construction for union

\( M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1) \) and \( M_2 = (Q_1, \Sigma, \delta_2, s_2, A_2) \)

Create \( M = (Q, \Sigma, \delta, s, A) \) where

- \( Q = Q_1 \times Q_2 = \{ (q_1, q_2) \mid q_1 \in Q_1, q_2 \in Q_2 \} \)
- \( s = (s_1, s_2) \)
- \( \delta : Q \times \Sigma \to Q \) where

\[
\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))
\]

- \( A = \{ (q_1, q_2) \mid q_1 \in A_1 \text{ or } q_2 \in A_2 \} = (A_1 \times Q_2) \cup (Q_1 \times A_2) \)

Theorem

\( L(M) = L(M_1) \cup L(M_2). \)
Set Difference

**Theorem**

\( M_1, M_2 \) DFAs. There is a DFA \( M \) such that

\[ L(M) = L(M_1) \setminus L(M_2). \]

**Exercise:** Prove the above using two methods.

- Using a direct product construction
- Using closure under complement and intersection and union

\[ L_1 - L_2 = L_1 \cup \overline{L_2} \]
**Question:** Why are DFAs required to only move right? Can we allow DFA to scan back and forth? *Caveat:* Tape is read-only so only memory is in machine’s state.
Questions: Why are DFAs required to only move right? Can we allow DFA to scan back and forth? Caveat: Tape is read-only so only memory is in machine’s state.

- Can define a formal notion of a “2-way” DFA
- Can show that any language recognized by a 2-way DFA can be recognized by a regular (1-way) DFA
- Proof is tricky simulation via NFAs