Here are several problems that are easy to solve in \( O(n) \) time, essentially by brute force. Your task is to design algorithms for these problems that are significantly faster.

1. Suppose we are given an array \( A[1..n] \) of \( n \) distinct integers, which could be positive, negative, or zero, sorted in increasing order so that \( A[1] < A[2] < \cdots < A[n] \).

   (a) Describe a fast algorithm that either computes an index \( i \) such that \( A[i] = i \) or correctly reports that no such index exists.

   **Solution:** Suppose we define a second array \( B[1..n] \) by setting \( B[i] = A[i] - i \) for all \( i \). For every index \( i \) we have

   \[
   \]

   so this new array is sorted in increasing order. Clearly, \( A[i] = i \) if and only if \( B[i] = 0 \). So we can find an index \( i \) such that \( A[i] = i \) by performing a binary search in \( B \). We don’t actually need to compute \( B \) in advance; instead, whenever the binary search needs to access some value \( B[i] \), we can just compute \( A[i] - i \) on the fly instead!

   Here are two formulations of the resulting algorithm, first recursive (keeping the array \( A \) as a global variable), and second iterative.

   ```
   (Return any index \( i \) such that \( \ell \leq i \leq r \) and \( A[i] = i \))
   FINDMATCH(\( \ell \), \( r \)):
   if \( \ell > r \)
      return None
   mid ← (\( \ell + r \))/2
   if \( A[mid] = mid \) \( \langle B[mid] = 0 \rangle \)
      return mid
   else if \( A[mid] < mid \) \( \langle B[mid] < 0 \rangle \)
      return FINDMATCH(mid + 1, \( r \))
   else \( \langle B[mid] > 0 \rangle \)
      return FINDMATCH(\( \ell \), mid − 1)
   
   FINDMATCH(\( A[1..n] \)):
   hi ← \( n \)
   lo ← 1
   while lo ≤ hi
      mid ← (lo + hi)/2
      if \( A[mid] = mid \) \( \langle B[mid] = 0 \rangle \)
         return mid
      else if \( A[mid] < mid \) \( \langle B[mid] < 0 \rangle \)
         lo ← mid + 1
      else \( \langle B[mid] > 0 \rangle \)
         hi ← mid − 1
   return None
   ```

   In both formulations, the algorithm is binary search, so it runs in \( O(\log n) \) time.
(b) Suppose we know in advance that $A[1] > 0$. Describe an even faster algorithm that either computes an index $i$ such that $A[i] = i$ or correctly reports that no such index exists. [Hint: This is really easy.]

**Solution:** The following algorithm solves this problem in $O(1)$ time:

```plaintext
FindMatchPos(A[1..n]):
  if A[1] = 1
    return 1
  else
    return None
```


For example, there are exactly six local minima in the following array:

```
9 7 7 2 1 3 7 5 4 7 3 3 4 8 6 9
```

Describe and analyze a fast algorithm that returns the index of one local minimum. For example, given the array above, your algorithm could return the integer 9, because $A[9]$ is a local minimum. [Hint: With the given boundary conditions, any array **must** contain at least one local minimum. Why?]

**Solution:** The following algorithm solves this problem in $O(\log n)$ time:

```plaintext
LOCAL_MIN(A[1..n]):
    if $n < 100$
        find the smallest element in A by brute force
        $m \leftarrow \lceil n/2 \rceil$
        if $A[m] < A[m+1]$
            return LOCAL_MIN(A[1..m+1])
        else
            return LOCAL_MIN(A[m..n])
```

If $n$ is less than 100, then a brute-force search runs in $O(1)$ time. There’s nothing special about 100 here; any other constant will do.

Otherwise, if $A[n/2] < A[n/2 + 1]$, the subarray $A[1..n/2 + 1]$ satisfies the precise boundary conditions of the original problem, so the recursion fairy will find local minimum inside that subarray.

Finally, if $A[n/2] > A[n/2 + 1]$, the subarray $A[n/2..n]$ satisfies the precise boundary conditions of the original problem, so the recursion fairy will find local minimum inside that subarray.

The running time satisfies the recurrence $T(n) \leq T(\lceil n/2 \rceil + 1) + O(1)$. Except for the +1 and the ceiling in the recursive argument, which we can ignore, this is the binary search recurrence, whose solution is $T(n) = O(\log n)$.

Alternatively, we can observe that $\lceil n/2 \rceil + 1 < 2n/3$ when $n \geq 100$, and therefore $T(n) \leq T(2n/3) + O(1)$, which implies $T(n) = O(\log_{3/2} n) = O(\log n)$. ■
3. Suppose you are given two sorted arrays \( A[1..n] \) and \( B[1..n] \) containing distinct integers. Describe a fast algorithm to find the median (meaning the \( n \)th smallest element) of the union \( A \cup B \). For example, given the input

\[
A[1..8] = [0, 1, 6, 9, 12, 13, 18, 20] \quad B[1..8] = [2, 4, 5, 8, 17, 19, 21, 23]
\]

your algorithm should return the integer 9. [Hint: What can you learn by comparing one element of \( A \) with one element of \( B \)?]

**Solution:** The following algorithm solves this problem in \( O(\log n) \) time:

```plaintext
MEDIAN(A[1..n], B[1..n]):
if n < 10^10
    use brute force
else if A[n/2] > B[n/2]
    return MEDIAN(A[1..n/2], B[n/2 + 1..n])
else
    return MEDIAN(A[n/2 + 1..n], B[1..n/2])
```

Suppose \( A[n/2] > B[n/2] \). Then \( A[n/2 + 1] \) is larger than all \( n \) elements in \( A[1..n/2] \cup B[1..n/2] \), and therefore larger than the median of \( A \cup B \), so we can discard the upper half of \( A \). Similarly, \( B[n/2 - 1] \) is smaller than all \( n + 1 \) elements of \( A[n/2..n] \cup B[n/2 + 1..n] \), and therefore smaller than the median of \( A \cup B \), so we can discard the lower half of \( B \). Because we discard the same number of elements from each array, the median of the remaining subarrays is the median of the original \( A \cup B \).

**To think about later:**

4. Now suppose you are given two sorted arrays \( A[1..m] \) and \( B[1..n] \) and an integer \( k \). Describe a fast algorithm to find the \( k \)th smallest element in the union \( A \cup B \). For example, given the input

\[
A[1..8] = [0, 1, 6, 9, 12, 13, 18, 20] \quad B[1..5] = [2, 5, 7, 17, 19] \quad k = 6
\]

your algorithm should return the integer 7.

**Solution:** The following algorithm solves this problem in \( O(\log \min\{k, m + n - k\}) = O(\log(m + n)) \) time:

```plaintext
SELECT(A[1..m], B[1..n], k):
if k < (m + n)/2
    return MEDIAN(A[1..k], B[1..k])
else
    return MEDIAN(A[k - n..m], B[k - m..n])
```

Here, \( \text{MEDIAN} \) is the algorithm from problem 3 with one minor tweak. If \( \text{MEDIAN} \) wants an entry in either \( A \) or \( B \) that is outside the bounds of the original arrays, it uses the value \(-\infty\) if the index is too low, or \(\infty\) if the index is too high, instead of creating a core dump.