Prove that each of the following languages is not regular.

1. \{0^{2^n} \mid n \geq 0\}

**Solution (verbose):** Let \( F = L = \{0^{2^n} \mid n \geq 0\} \).

Let \( x \) and \( y \) be arbitrary elements of \( F \).

Then \( x = 0^{2^i} \) and \( y = 0^{2^j} \) for some non-negative integers \( x \) and \( y \).

Let \( z = 0^{2^i} \).

Then \( xz = 0^{2^i}0^{2^i} = 0^{2^i+1} \in L \).

And \( yz = 0^{2^i}0^{2^j} = 0^{2^i+2^j} \not\in L \), because \( i \neq j \) 

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular. \( \blacksquare \)

**Solution (concise):** For any non-negative integers \( i \neq j \), the strings \( 0^{2^i} \) and \( 0^{2^j} \) are distinguished by the suffix \( 0^{2^i} \), because \( 0^{2^i}0^{2^i} = 0^{2^i+1} \in L \) but \( 0^{2^i}0^{2^j} = 0^{2^i+2^j} \not\in L \). Thus \( L \) itself is an infinite fooling set for \( L \). \( \blacksquare \)

2. \{0^{2n}1^n \mid n \geq 0\}

**Solution (verbose):** Let \( F \) be the language \( 0^* \).

Let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Let \( z = 0^i1^i \).

Then \( xz = 0^{2i}1^i \in L \).

And \( yz = 0^{i+j}1^i \not\in L \), because \( i+j \neq 2i \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular. \( \blacksquare \)

**Solution (concise):** For all non-negative integers \( i \neq j \), the strings \( 0^i \) and \( 0^j \) are distinguished by the suffix \( 0^i1^i \), because \( 0^{2i}1^i \in L \) but \( 0^{i+j}1^i \not\in L \). Thus, the language \( 0^* \) is an infinite fooling set for \( L \). \( \blacksquare \)

**Solution (concise, different fooling set):** For all non-negative integers \( i \neq j \), the strings \( 0^{2i} \) and \( 0^{2j} \) are distinguished by the suffix \( 1^i \), because \( 0^{2i}1^i \in L \) but \( 0^{2j}1^i \not\in L \). Thus, the language \((00)^* \) is an infinite fooling set for \( L \). \( \blacksquare \)
3. \(\{\theta^m 1^n \mid m \neq 2n\}\)

Solution (verbose): Let \(F\) be the language \(\theta^*\).
Let \(x\) and \(y\) be arbitrary strings in \(F\).
Then \(x = \theta^i\) and \(y = \theta^j\) for some non-negative integers \(i \neq j\).
Let \(z = \theta^i 1^i\).
Then \(xz = \theta^{2i} 1^i \notin L\).
And \(yz = \theta^{i+j} 1^i \notin L\), because \(i + j \neq 2i\).
Thus, \(F\) is a fooling set for \(L\).
Because \(F\) is infinite, \(L\) cannot be regular.

Solution (concise, different fooling set): For all non-negative integers \(i \neq j\), the strings \(\theta^{2i}\) and \(\theta^{2j}\) are distinguished by the suffix \(1^i\), because \(\theta^{2i} 1^i \notin L\) but \(\theta^{2j} 1^i \notin L\). Thus, the language \((\theta \theta)^*\) is an infinite fooling set for \(L\).

4. Strings over \(\{\theta, 1\}\) where the number of \(\theta\)s is exactly twice the number of \(1\)s.

Solution (verbose): Let \(F\) be the language \(\theta^*\).
Let \(x\) and \(y\) be arbitrary strings in \(F\).
Then \(x = \theta^i\) and \(y = \theta^j\) for some non-negative integers \(i \neq j\).
Let \(z = \theta^i 1^i\).
Then \(xz = \theta^{2i} 1^i \notin L\).
And \(yz = \theta^{i+j} 1^i \notin L\), because \(i + j \neq 2i\).
Thus, \(F\) is a fooling set for \(L\).
Because \(F\) is infinite, \(L\) cannot be regular.

Solution (concise, different fooling set): For all non-negative integers \(i \neq j\), the strings \(\theta^{2i}\) and \(\theta^{2j}\) are distinguished by the suffix \(1^i\), because \(\theta^{2i} 1^i \notin L\) but \(\theta^{2j} 1^i \notin L\). Thus, the language \((\theta \theta)^*\) is an infinite fooling set for \(L\).

Solution (closure properties): If \(L\) were regular, then the language
\[
((\theta + 1)^* \setminus L) \cap \theta^* 1^* = \{\theta^m 1^n \mid m \neq 2n\}
\]
would also be regular, because regular languages are closed under complement and intersection. But we just proved that \(\{\theta^m 1^n \mid m \neq 2n\}\) is not regular in problem 3. [Yes, this proof would be worth full credit, either in homework or on an exam.]
5. Strings of properly nested parentheses (), brackets [], and braces {}. For example, the string ([[]]){} is in this language, but the string [[]] is not, because the left and right delimiters don’t match.

**Solution (verbose):** Let $F$ be the language $^*$. Let $x$ and $y$ be arbitrary strings in $F$. Then $x = (i)$ and $y = (j)$ for some non-negative integers $i \neq j$. Let $z = )^i$. Then $xz = (i)^i \in L$. And $yz = (j)^i \notin L$, because $i \neq j$.

Thus, $F$ is a fooling set for $L$. Because $F$ is infinite, $L$ cannot be regular.

**Solution (concise):** For any non-negative integers $i \neq j$, the strings $(i)$ and $(j)$ are distinguished by the suffix $)jl$, because $(i)^i \in L$ but $(j)^j \notin L$. Thus, the language $^*$ is an infinite fooling set.

6. Strings of the form $w_1 \# w_2 \# \cdots \# w_n$ for some $n \geq 2$, where each substring $w_i$ is a string in $\{0, 1\}^*$, and some pair of substrings $w_i$ and $w_j$ are equal.

**Solution (verbose):** Let $F$ be the language $0^*$. Let $x$ and $y$ be arbitrary strings in $F$. Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$. Let $z = \#0^i$. Then $xz = 0^i\#0^i \in L$. And $yz = 0^j\#0^i \notin L$, because $i \neq j$.

Thus, $F$ is a fooling set for $L$. Because $F$ is infinite, $L$ cannot be regular.

**Solution (concise):** For any non-negative integers $i \neq j$, the strings $0^i$ and $0^j$ are distinguished by the suffix $\#0^i$, because $0^i\#0^i \in L$ but $0^j\#0^i \notin L$. Thus, the language $0^*$ is an infinite fooling set.
Work on these later:

7. $\{\theta^{n^2} \mid n \geq 0\}$

**Solution:** Let $x$ and $y$ be distinct arbitrary strings in $L$.
Without loss of generality, $x = \theta^{i^2}$ and $y = \theta^{j^2}$ for some $j > i \geq 0$.
Let $z = \theta^{2i+1}$.
Then $xz = \theta^{i^2+2i+1} = \theta^{(i+1)^2} \in L$.
On the other hand, $yz = \theta^{j^2+2i+1} \notin L$, because $j^2 < j^2 + 2i + 1 < j^2 + 2j + 1 = (j+1)^2$.
Thus, $z$ distinguishes $x$ and $y$.
We conclude that $L$ is an infinite fooling set for $L$, so $L$ cannot be regular.

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Solution: Let $x$ and $y$ be distinct arbitrary strings in $\theta^*$. 
Without loss of generality, $x = \theta^i$ and $y = \theta^j$ for some $i > j \geq 0$.
Let $z = \theta^{i^2+i+1}$.
Then $xz = \theta^{i^2+2i+1} = \theta^{(i+1)^2} \in L$.
On the other hand, $yz = \theta^{j^2+i+j+1} \notin L$, because $i^2 < i^2 + i + j + 1 < (i+1)^2$.
Thus, $z$ distinguishes $x$ and $y$.
We conclude that $\theta^*$ is an infinite fooling set for $L$, so $L$ cannot be regular.

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Solution: Let $x$ and $y$ be distinct arbitrary strings in $0000^*$. 
Without loss of generality, $x = \theta^i$ and $y = \theta^j$ for some $i > j \geq 3$.
Let $z = \theta^{i^2-i}$.
Then $xz = \theta^{i^2} \in L$.
On the other hand, $yz = \theta^{i^2-i+j} \notin L$, because
\[(i-1)^2 = i^2 - 2i + 1 < i^2 - i < i^2 - i + j < i^2.\]
(The first inequalities requires $i \geq 2$, and the second $j \geq 1$.)
Thus, $z$ distinguishes $x$ and $y$.
We conclude that $0000^*$ is an infinite fooling set for $L$, so $L$ cannot be regular.
8. \( \{w \in (0 + 1)^* \mid w \text{ is the binary representation of a perfect square}\} \)

**Solution:** We design our fooling set around numbers of the form \((2^k + 1)^2 = 2^{2k} + 2^{k+1} + 1 = 10^{k-2}10^k1 \in L\), for any integer \(k \geq 2\). The argument is somewhat simpler if we further restrict \(k\) to be even.

Let \(F = 1(00)^*1\), and let \(x\) and \(y\) be arbitrary strings in \(F\).

Then \(x = 10^{2i-2}1\) and \(y = 10^{2j-2}1\), for some positive integers \(i \neq j\).

Without loss of generality, assume \(i < j\). (Otherwise, swap \(x\) and \(y\).)

Let \(z = 0^{2i}1\).

Then \(xz = 10^{2i-2}10^{2j}1\) is the binary representation of \(2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2\), and therefore \(xz \in L\).

On the other hand, \(yz = 10^{2j-2}10^{2i}1\) is the binary representation of \(2^{2i+2j} + 2^{2i+1} + 1\). Simple algebra gives us the inequalities

\[
(2^{i+j})^2 = 2^{2i+2j} < 2^{2i+2j} + 2^{2i+1} + 1 < 2^{2(i+j)} + 2^{i+j+1} + 1 = (2^{i+j} + 1)^2.
\]

So \(2^{2i+2j} + 2^{2i+1} + 1\) lies between two consecutive perfect squares, and thus is not a perfect square, which implies that \(yz \notin L\).

We conclude that \(F\) is a fooling set for \(L\). Because \(F\) is infinite, \(L\) cannot be regular. ■