Prove that each of the following languages is **not** regular.

1. \( \{ \theta^{2^n} \mid n \geq 0 \} \)

   **Solution:** For any non-negative integers \( i \neq j \), the strings \( \theta^{2^i} \) and \( \theta^{2^j} \) are distinguished by the suffix \( \theta^{2^i} \). Thus \( L \) itself is an infinite fooling set for \( L \).

2. \( \{ \theta^{2^n} 1^n \mid n \geq 0 \} \)

   **Solution:** For all non-negative integers \( i \neq j \), the strings \( \theta^{2^i} \) and \( \theta^{2^j} \) are distinguished by the suffix \( 1^i \). Thus, the language \( (\theta\theta)^* \) is an infinite fooling set.

   **Solution:** For all non-negative integers \( i \neq j \), the strings \( \theta^i \) and \( \theta^j \) are distinguished by the suffix \( \theta^i 1^i \). Thus, the language \( \theta^* \) is an infinite fooling set.
3. \( \{0^m1^n \mid m \neq 2n\} \)

**Solution:** For all non-negative integers \( i \neq j \), the strings \( 0^{2i} \) and \( 0^{2j} \) are distinguished by the suffix \( 1^i \). Thus, the language \((00)^*\) is an infinite fooling set. ■

**Solution:** For all non-negative integers \( i \neq j \), the strings \( 0^i \) and \( 0^j \) are distinguished by the suffix \( 0^i1^i \). Thus, the language \( 0^* \) is an infinite fooling set. ■

4. Strings over \( \{0, 1\} \) where the number of 0s is exactly twice the number of 1s.

**Solution:** For all non-negative integers \( i \neq j \), the strings \( 0^i \) and \( 0^j \) are distinguished by the suffix \( 0^i1^i \). Thus, the language \( 0^* \) is an infinite fooling set. ■

5. Strings of properly nested parentheses ( ), brackets [ ], and braces { }. For example, the string \( [[[\ldots]]] \) is in this language, but the string \( [\ldots] \) is not, because the left and right delimiters don’t match.

**Solution:** For any non-negative integers \( i \neq j \), the strings \( (^i \) and \( (^j \) are distinguished by the suffix \( )^i \). Thus, the language \( (^* \) is an infinite fooling set. ■

6. Strings of the form \( w_1\#w_2\#\cdots\#w_n \) for some \( n \geq 2 \), where each substring \( w_i \) is a string in \( \{0, 1\}^* \), and some pair of substrings \( w_i \) and \( w_j \) are equal.

**Solution:** For any non-negative integers \( i \neq j \), the strings \( 0^i \) and \( 0^j \) are distinguished by the suffix \( \#0^i \). Thus, the language \( 0^* \) is an infinite fooling set. ■

**Work on these later:**

7. \( \{0^n \mid n \geq 0\} \)

**Solution:** Let \( x \) and \( y \) be distinct arbitrary strings in \( L \).

Without loss of generality, \( x = 0^{i^2} \) and \( y = 0^{j^2} \) for some \( i > j \geq 0 \).

Let \( z = 0^{2i+1} \).

Then \( xz = 0^{2^2+2i+1} = 0^{(i+1)^2} \in L \)

On the other hand, \( yz = 0^{2^2+2j+1} \notin L \), because \( i^2 < i^2 + 2j + 1 < (i + 1)^2 \).

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( L \) is an infinite fooling set for \( L \), so \( L \) cannot be regular. ■
Solution: Let $x$ and $y$ be distinct arbitrary strings in $0^*$. Without loss of generality, $x = 0^i$ and $y = 0^j$ for some $i > j \geq 0$. Let $z = 0^{i^2+i+1}$. Then $xz = 0^{i^2+2i+1} = 0^{(i+1)^2} \in L$. On the other hand, $yz = 0^{i^2+i+j+1} \notin L$, because $i^2 < i^2 + i + j + 1 < (i+1)^2$. Thus, $z$ distinguishes $x$ and $y$. We conclude that $0^*$ is an infinite fooling set for $L$, so $L$ cannot be regular.

Solution: Let $x$ and $y$ be distinct arbitrary strings in $0000^*$. Without loss of generality, $x = 0^i$ and $y = 0^j$ for some $i > j \geq 3$. Let $z = 0^{i^2-i}$. Then $xz = 0^{i^2} \in L$. On the other hand, $yz = 0^{i^2-i+j} \notin L$, because 

\[
(i-1)^2 = i^2 - 2i + 1 < i^2 - i < i^2 - i + j < i^2.
\]

(The first inequalities requires $i \geq 2$, and the second $j \geq 1$.) Thus, $z$ distinguishes $x$ and $y$. We conclude that $0^*$ is an infinite fooling set for $L$, so $L$ cannot be regular.

8. \{w \in (0+1)^* \mid w \text{ is the binary representation of a perfect square}\}

Solution: We design our fooling set around numbers of the form $(2^k+1)^2 = 2^{2k} + 2^{k+1} + 1 = 10^{k-2}010^k1 \in L$, for any integer $k \geq 2$. The argument is somewhat simpler if we further restrict $k$ to be even.

Let $x$ and $y$ be arbitrary strings in the language $1(00)^*1$. Without loss of generality, $x = 10^{2i-2}1$ and $y = 10^{2j-2}1$, for some positive integers $i < j$. Let $z = 0^{2i}1$. Then $xz = 10^{2i-2}10^{2j}1$ is the binary representation of $2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2$, and therefore $xz \in L$. On the other hand, $yz = 10^{2j-2}10^{2j}1$ is the binary representation of $2^{2i+2j} + 2^{2i+1} + 1$. We immediately have

\[
(2^{i+j})^2 = 2^{2i+2j} < 2^{2i+2j} + 2^{2i+1} + 1 < 2^{2(i+j)} + 2^{i+j+1} + 1 = (2^{i+j} + 1)^2.
\]

So $2^{2i+2j} + 2^{2i+1} + 1$ is not a perfect square, and therefore $yz \notin L$.

We conclude that $1(00)^*1$ is an infinite fooling set for $L$, so $L$ is not regular.