The following problems ask you to prove some “obvious” claims about recursively-defined string functions. In each case, we want a self-contained, step-by-step induction proof that builds on formal definitions and prior results, not on intuition. In particular, your proofs must refer to the formal recursive definitions of string length and string concatenation:

$$\begin{align*}
|w| &:= \begin{cases} 
0 & \text{if } w = \epsilon \\
1 + |x| & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x
\end{cases} \\
w \cdot z &:= \begin{cases} 
z & \text{if } w = \epsilon \\
a \cdot (x \cdot z) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x
\end{cases}
\end{align*}$$

You may freely use the following results, which were proved in the lecture notes:

**Lemma 1:** $w \cdot \epsilon = w$ for all strings $w$.

**Lemma 2:** $|w \cdot x| = |w| + |x|$ for all strings $w$ and $x$.

**Lemma 3:** $(w \cdot x) \cdot y = w \cdot (x \cdot y)$ for all strings $w$, $x$, and $y$.

The reversal $w^R$ of a string $w$ is defined recursively as follows:

$$w^R := \begin{cases}
\epsilon & \text{if } w = \epsilon \\
x^R \cdot a & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x
\end{cases}$$

For example, $\text{STRESSED}^R = \text{DESSERTS}$ and $\text{WTF374}^R = \text{473FTW}$.

1. Prove that $|w^R| = |w|$ for every string $w$.

**Solution (induction on $w$):**

Let $w$ be an arbitrary string.

Assume for any string $x$ where $|x| < |w|$ that $|x^R| = |x|$.

There are two cases to consider.

- If $w = \epsilon$, then

  $$|w^R| = |\epsilon| \quad \text{by definition of } w^R$$

  $$= |w| \quad \text{by definition of } |.|$$

- Otherwise, $w = ax$ for some symbol $a$ and some string $x$. In that case, we have

  $$|w^R| = |x^R \cdot a| \quad \text{by definition of } w^R$$

  $$= |x^R| + |a| \quad \text{by Lemma 2}$$

  $$= |x^R| + 1 \quad \text{by definition of } |.| \text{ (twice)}$$

  $$= |x| + 1 \quad \text{by the induction hypothesis} = |w| \quad \text{by definition of } |.|$$

In both cases, we conclude that $|w^R| = |w|$. ■
2. Prove that \((w \cdot z)^R = z^R \cdot w^R\) for all strings \(w\) and \(z\).

**Solution (induction on \(w\)):**

Let \(w\) and \(z\) be arbitrary strings.

Assume for any string \(x\) where \(|x| < |w|\) that \((x \cdot z)^R = z^R \cdot x^R\).

There are two cases to consider:

- If \(w = \epsilon\), then
  \[
  (w \cdot z)^R = z^R \cdot \epsilon = z^R \cdot w^R
  \]
  by definition of \(\cdot\) and Lemma 1.

- Otherwise, \(w = ax\) for some symbol \(a\) and some string \(x\).
  \[
  (w \cdot z)^R = (a \cdot (x \cdot z))^R
  \]
  by definition of \(\cdot\)
  \[
  = (x \cdot z)^R \cdot a
  \]
  by definition of \(\cdot^R\)
  \[
  = (z^R \cdot x^R)^R \cdot a
  \]
  by the induction hypothesis, because \(|x| < |w|\)
  \[
  = z^R \cdot (x^R \cdot a)
  \]
  by Lemma 3
  \[
  = z^R \cdot w^R
  \]
  by definition of \(\cdot^R\).

In both cases, we conclude that \((w \cdot z)^R = z^R \cdot w^R\). \(\blacksquare\)

But how did I know that the induction hypothesis needs to change the first string \(w\), but not the second string \(z\)? I wrote down the inductive argument first, and then noticed that in the proof for \(w \cdot z\), we needed the inductive hypothesis on \(x \cdot z\). Same string \(z\), but \(w\) changed to \(x\). Alternatively, in light of Lemma 2, I could have inducted on the **sum** of the string lengths with the inductive hypothesis "Assume for all strings \(x\) and \(y\) such that \(|x| + |y| < |w| + |z|\) that \((x \cdot y)^R = y^R \cdot x^R\)."

3. Prove that \((w^R)^R = w\) for every string \(w\).

**Solution (induction on \(w\)):**

Let \(w\) be an arbitrary string.

Assume for any string \(x\) where \(|x| < |w|\) that \((x^R)^R = x\).

There are two cases to consider.

- If \(w = \epsilon\), then \((w^R)^R = \epsilon^R = \epsilon\) by definition.

- Otherwise, \(w = ax\) for some symbol \(a\) and some string \(x\).
  \[
  (w^R)^R = (x^R \cdot a)^R
  \]
  by definition of \(\cdot^R\)
  \[
  = a^R \cdot (x^R)^R
  \]
  by problem 2
  \[
  = a^R \cdot (x^R)^R
  \]
  by definition of \(\cdot^R\)
  \[
  = a \cdot (x^R)^R
  \]
  by definition of \(\cdot\)
  \[
  = a \cdot x
  \]
  by the induction hypothesis
  \[
  = w
  \]
  by assumption

In both cases, we conclude that \((w^R)^R = w\). \(\blacksquare\)
To think about later: Let \( (a, w) \) denote the number of times symbol \( a \) appears in string \( w \). For example, \( (X, \text{WTF374}) = 0 \) and \( (0, \text{0000101010101010100}) = 12 \).

4. Give a formal recursive definition of \( (a, w) \).

Solution:

\[
(a, w) = \begin{cases} 
0 & \text{if } w = \varepsilon \\
1 + (a, x) & \text{if } w = ax \text{ for some string } x \\
(a, x) & \text{if } w = bx \text{ for some symbol } b \neq a \text{ and some string } x
\end{cases}
\]

5. Prove that \( (a, w \cdot z) = (a, w) + (a, z) \) for all symbols \( a \) and all strings \( w \) and \( z \).

Solution (induction on \( w \)):

Let \( a \) be an arbitrary symbol, and let \( w \) and \( z \) be arbitrary strings.

Assume for any string \( x \) such that \( |x| < |w| \) that \( (a, x \cdot z) = (a, x) + (a, z) \).

There are three cases to consider.

- If \( w = \varepsilon \), then
  \[
  (a, w \cdot x) = (a, x) \quad \text{by definition of } \cdot
  \]
  \[
  = (a, w) + (a, x) \quad \text{by definition of } #
  \]

- If \( w = ax \) for some string \( x \), then
  \[
  (a, w \cdot z) = (a, a \cdot x \cdot z)
  \]
  \[
  = (a, a \cdot (x \cdot z)) \quad \text{by definition of } \cdot
  \]
  \[
  = 1 + (a, x \cdot z) \quad \text{by definition of } #
  \]
  \[
  = 1 + (a, x) + (a, z) \quad \text{by the induction hypothesis}
  \]
  \[
  = (a, ax) + (a, z) \quad \text{by definition of } #
  \]
  \[
  = (a, w) + (a, z) \quad \text{because } w = ax
  \]

- If \( w = bx \) for some symbol \( b \neq a \) and some string \( x \), then
  \[
  (a, w \cdot z) = (a, b \cdot (x \cdot z)) \quad \text{by definition of } \cdot
  \]
  \[
  = (a, x \cdot z) \quad \text{by definition of } #
  \]
  \[
  = (a, x) + (a, z) \quad \text{by the induction hypothesis}
  \]
  \[
  = (a, bx) + (a, z) \quad \text{by definition of } #
  \]
  \[
  = (a, w) + (a, z) \quad \text{because } w = bx
  \]

In every case, we conclude that \( (a, w \cdot z) = (a, w) + (a, z) \).
6. Prove that \( \#(a, w^R) = \#(a, w) \) for all symbols \( a \) and all strings \( w \).

**Solution (induction on \( w \)):** Let \( a \) be an arbitrary symbol, and let \( w \) be an arbitrary string. Assume for any string \( x \) such that \( |x| < |w| \) that \( \#(a, x^R) = \#(a, x) \).

There are three cases to consider.

- If \( w = \epsilon \), then \( w^R = \epsilon = w \) by definition, so \( \#(a, w^R) = \#(a, w) \).
- If \( w = ax \) for some string \( x \), then
  \[
  \#(a, w^R) = \#(a, x^R \cdot a) \quad \text{by definition of } R
  = \#(a, x^R) + \#(a, a) \quad \text{by problem 5}
  = \#(a, x^R) + 1 \quad \text{by definition of } \#
  = \#(a, x) + 1 \quad \text{by the induction hypothesis}
  = \#(a, w) \quad \text{by definition of } \#
  
  
  \]

- If \( w = bx \) for some symbol \( b \neq a \) and some string \( x \), then
  \[
  \#(a, w^R) = \#(a, x^R \cdot b) \quad \text{by definition of } R
  = \#(a, x^R) + \#(a, b) \quad \text{by problem 5}
  = \#(a, x^R) \quad \text{by definition of } \#
  = \#(a, x) \quad \text{by the induction hypothesis}
  = \#(a, w) \quad \text{by definition of } \#
  
  
  
  
  
  
  In every case, we conclude that \( \#(a, w^R) = \#(a, w) \). \]