374: Lecture 26 11/29/2022

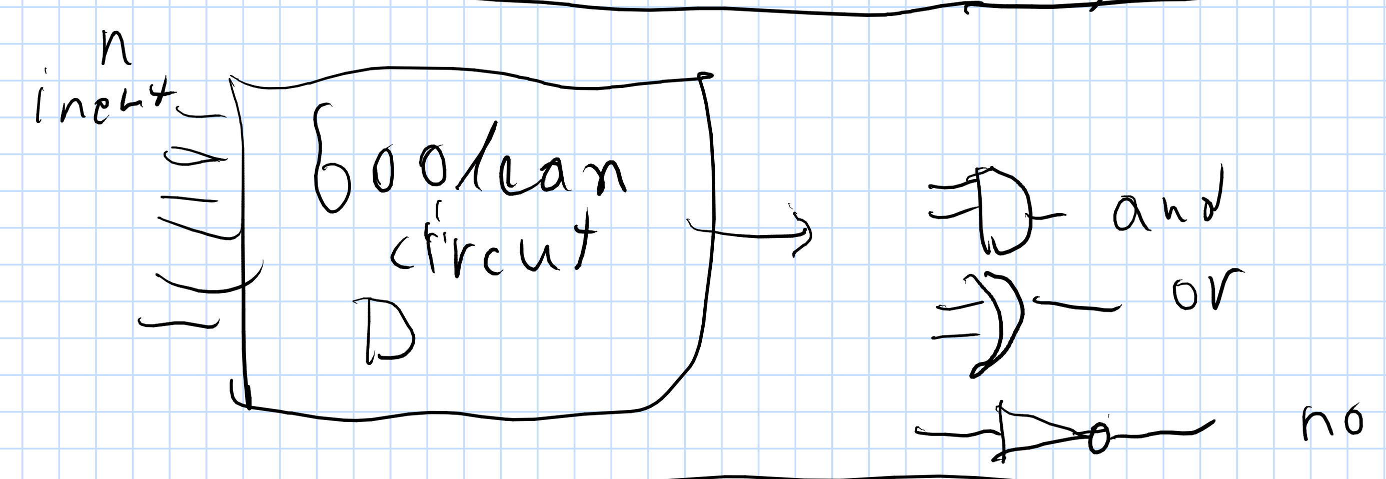
NP completeness

Cook-Levin theorem

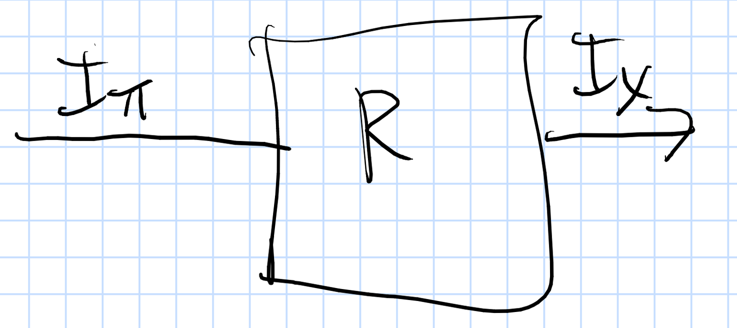
SAT is NP-Complete.

Def
A decision problem is NPC if:
- It is in NP
- If one can solve the problem in poly time $\Rightarrow$ all problems in NP can be solved in poly time $P = NP$.

CSAT: circuit satisfiability



How to prove that a problem X is NPC:
- Prove that it is in NP.
- Find a known NPC problem Π
  prove $\Pi \le_P X$ (show a poly reduction from Π to X.)



I_Π is TRUE $\Leftrightarrow$ I_X is TRUE.

$CSAT =_P SAT =_P 3SAT$ | 2SAT

3 COLOR

Input: $G = (V, E)$ undirected with n vertices
Q: Is G 3-colorable?

Lemma 3COLOR is in NP.

2 coloring?

A reduction from $3SAT \le_P 3COLOR$.

F an instance of 3SAT.

$(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\quad) \ldots$ $C_1, \ldots, C_m$ clauses; $x_1, \ldots, x_n$ variables

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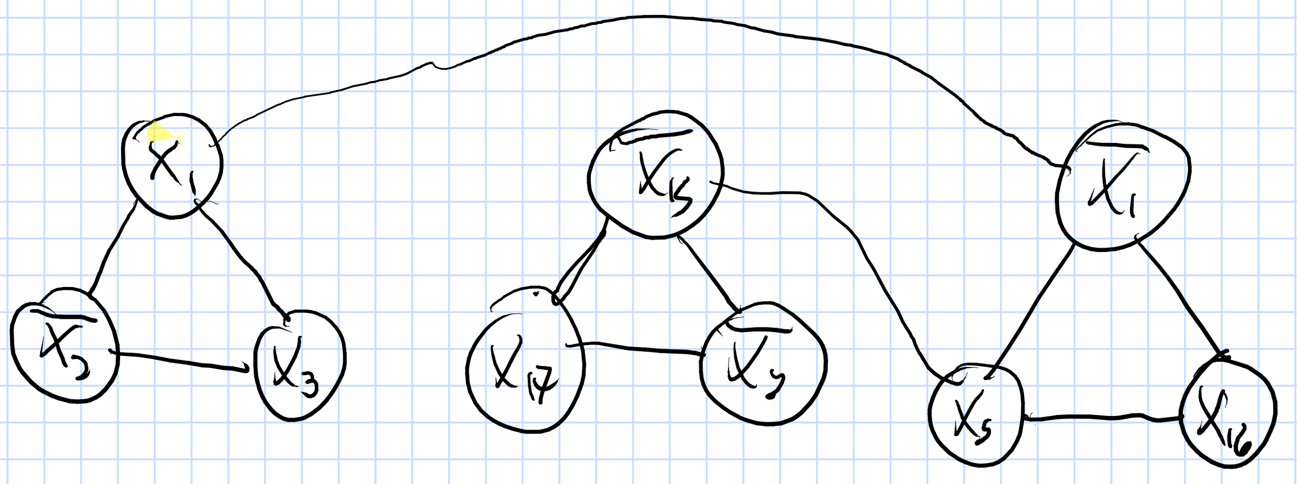
$3SAT \le_P$ INDEPENDENT SET.

$(x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_{17} \vee \bar{x}_5 \vee \bar{x}_{15})(\bar{x}_1 \vee x_5 \vee x_{16}) \ldots$



Claim Constructed graph has IS of size m (# of clauses in F)
$\Leftrightarrow$ The formula has a satisfying assignment.

proof
$\Rightarrow$ S be an IS in G of size m.
  $\Rightarrow$ S has exactly one vertex in each triangle that corresponds to a clause.
  $\Rightarrow$ interpret chosen vertices as an assignment.
  if $v \in S$ and v is labeled:
  - label(v) $= x_i \Rightarrow$ assign $x_i \leftarrow 1$
  - label(v) $= \bar{x}_i \Rightarrow$ assign $x_i \leftarrow 0$.

  This assignment is consistent
  1. No variable is assigned diff values. since S can not include two vertices with complemented label.
  2. It is a satisfying assignment.

$\Leftarrow$ A sat assignment for F induces an IS in G of size m. $\blacksquare$

- poly time reduction $3SAT \le IS$
- IS is in NP.

$\Rightarrow$ Independent set is NPC!

$IS =_P CLIQUE =_P VC$

$\Rightarrow$ CLIQUE and VC are NPC!