Algorithms & Models of Computation

CS/ECE 374, Fall 2020

NP and NP Completeness

Lecture 23

Tuesday, December 1, 2020

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23.1
NP-Completeness: Cook-Levin Theorem

23.1.1
Completeness

NP: Non-deterministic polynomial

Definition 23.1.

A decision problem is in **NP***, if it has a polynomial time certifier, for all the all the YES instances.*

Definition 23.2.

A decision problem is in **co-NP***, if it has a polynomial time certifier, for all the all the NO instances.*

Example 23.3.

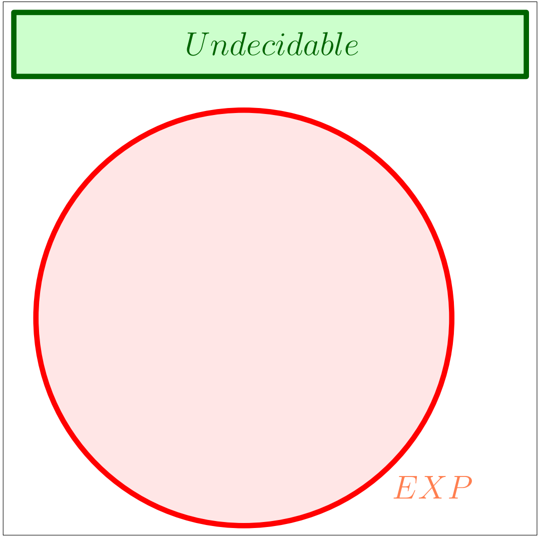
1. **3SAT** is in **NP**.
2. But **Not3SAT** is in **co-NP**.

In the beginning...

In the beginning...

Undecidable

In the beginning...



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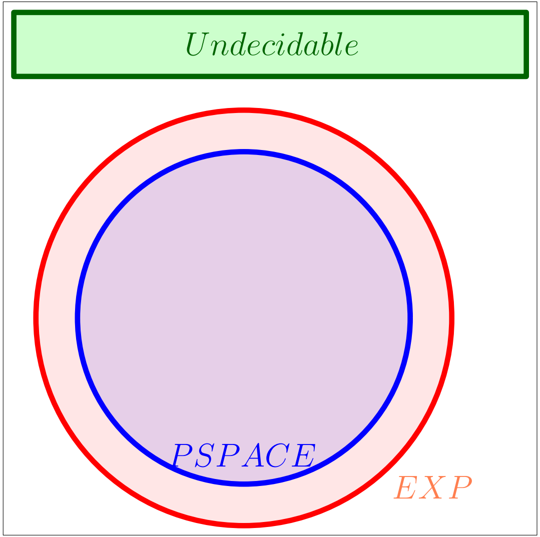

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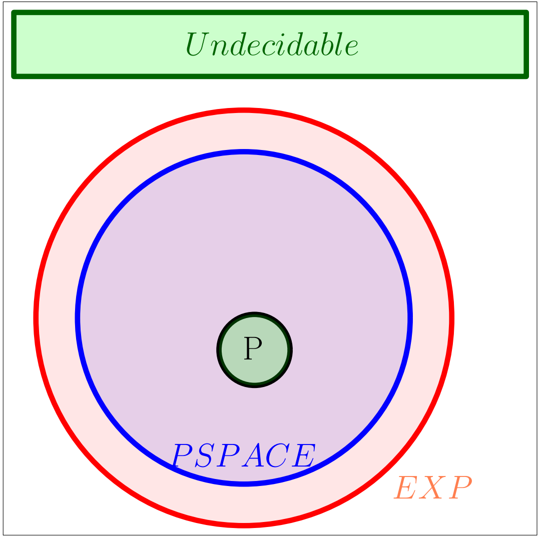

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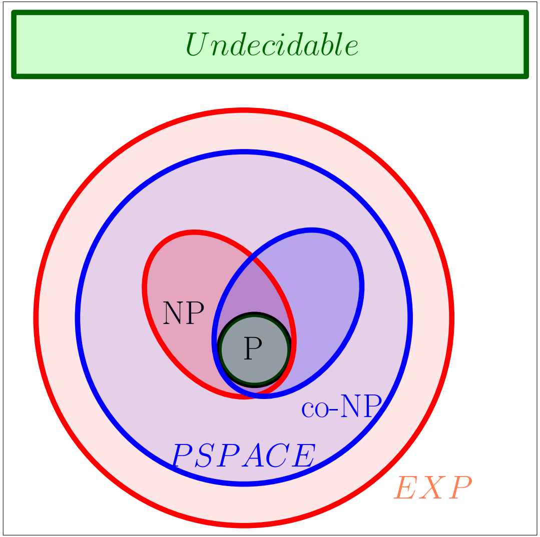

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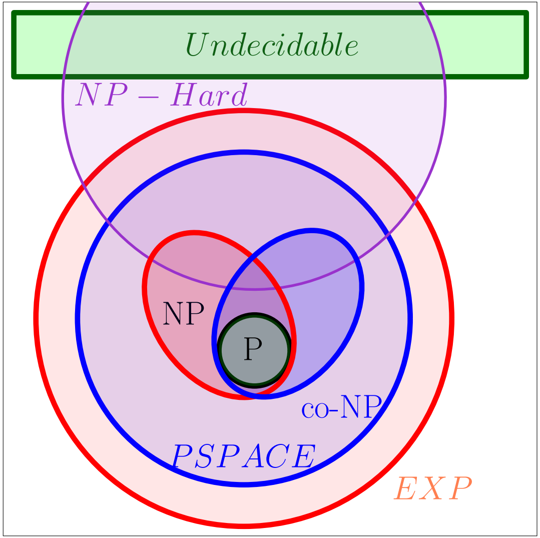

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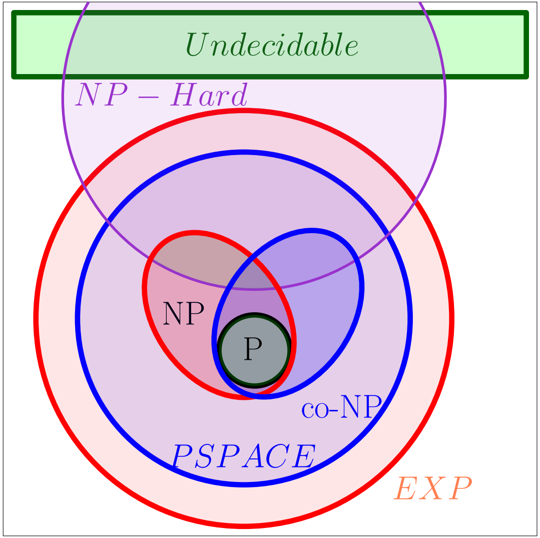

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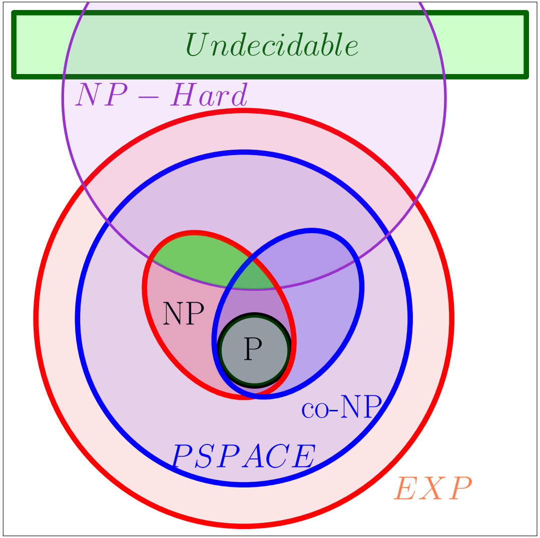

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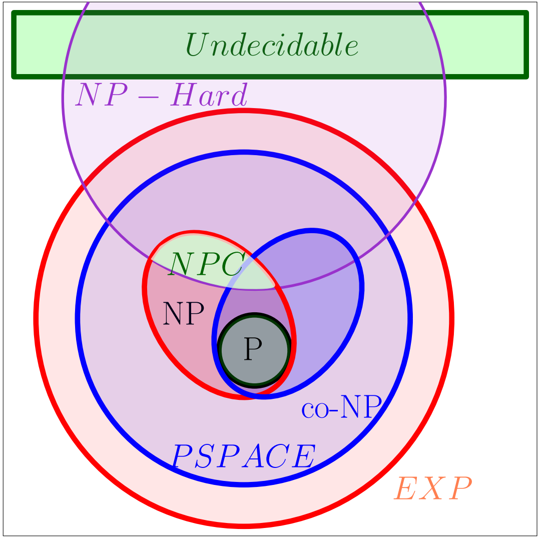

"Hardest" Problems

Question

What is the hardest problem in **NP**? How do we define it?

Towards a definition

1. Hardest problem must be in **NP**.
2. Hardest problem must be at least as "difficult" as every other problem in **NP**.

NP-Complete Problems

Definition 23.4.

A problem $\boldsymbol{X}$ is said to be **NP-Complete** if

1. $\boldsymbol{X} \in$ **NP**, and
2. (Hardness) For any $\boldsymbol{Y} \in$ **NP**, $\mathbf{Y} \leq_{\boldsymbol{P}} \mathbf{X}$.

Solving **NP-Complete** Problems

Proposition 23.5.

Suppose X is **NP-Complete**. *Then X can be solved in polynomial time* $\iff$ **P** $=$ **NP**.

Proof.

$\Rightarrow$ Suppose X can be solved in polynomial time

0.1 Let $Y \in$ **NP**. We know $Y \leq_P X$.

0.2 We showed that if $Y \leq_P X$ and X can be solved in polynomial time, then Y can be solved in polynomial time.

0.3 Thus, every problem $Y \in$ **NP** is such that $Y \in P$.

0.4 $\implies NP \subseteq P$.

0.5 Since **P** $\subseteq$ **NP**, we have **P** $=$ **NP**.

$\Leftarrow$ Since **P** $=$ **NP**, and $X \in$ **NP**, we have a polynomial time algorithm for X. $\square$

NP-Hard Problems

Definition 23.6.

A problem X is said to be **NP-Hard** if

1. (Hardness) For any $Y \in$ **NP**, we have that **Y** $\leq_P$ **X**.

An **NP-Hard** problem need not be in **NP**!

Example: Halting problem is **NP-Hard** (why?) but not **NP-Complete**.

Consequences of proving **NP-Complete**ness

If X is **NP-Complete**

1. Since we believe $\mathbf{P} \neq \mathbf{NP}$,
2. and solving X implies $\mathbf{P} = \mathbf{NP}$.

X is unlikely to be efficiently solvable.

At the very least, many smart people before you have failed to find an efficient algorithm for X.
(This is proof by mob opinion — take with a grain of salt.)

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THE END

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(for now)

23.1.2
SAT is NP-Complete

NP-Complete Problems

Question

Are there any problems that are **NP-Complete**?

Answer

Yes! Many, many problems are **NP-Complete**.

Cook-Levin Theorem

Theorem 23.7 (Cook-Levin).

SAT *is* **NP-Complete**.

Need to show

1. **SAT** is in **NP**.
2. every **NP** problem ***X*** reduces in polynomial time to **SAT**.

Might see proof later...

Steve Cook won the Turing award for his theorem.

Cook-Levin Theorem

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23.1.3
Other NP Complete Problems

Proving that a problem **X** is **NP-Complete**

To prove X is **NP-Complete**, show

1. Show that X is in **NP**.
2. Give a polynomial-time reduction from a known **NP-Complete** problem such as **SAT** to X

SAT $\leq_P X$ implies that every **NP** problem $Y \leq_P X$. Why?
Transitivity of reductions:

$Y \leq_P SAT$ and $SAT \leq_P X$ and hence $Y \leq_P X$.

Proving that a problem X is NP-Complete

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Proving that a problem **X** is **NP-Complete**

To prove $\boldsymbol{X}$ is **NP-Complete**, show

1. Show that $\boldsymbol{X}$ is in **NP**.
2. Give a polynomial-time reduction <u>from</u> a known **NP-Complete** problem such as **SAT** <u>to</u> $\boldsymbol{X}$

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Transitivity of reductions:

$\boldsymbol{Y} \leq_P \boldsymbol{SAT}$ and $\boldsymbol{SAT} \leq_P \boldsymbol{X}$ and hence $\boldsymbol{Y} \leq_P \boldsymbol{X}$.

3-SAT is NP-Complete

- **3-SAT** is in ***NP***
- **SAT** $\leq_P$ **3-SAT** as we saw

NP-Completeness via Reductions

1. **SAT** is **NP-Complete** due to Cook-Levin theorem
2. **SAT** $\leq_P$ **3-SAT**
3. **3-SAT** $\leq_P$ **Independent Set**
4. **Independent Set** $\leq_P$ **Vertex Cover**
5. **Independent Set** $\leq_P$ **Clique**
6. **3-SAT** $\leq_P$ **3-Color**
7. **3-SAT** $\leq_P$ **Hamiltonian Cycle**

Hundreds and thousands of different problems from many areas of science and engineering have been shown to be **NP-Complete**.

A surprisingly frequent phenomenon!

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THE END

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23.2
Reducing **3-SAT** to Independent Set

Independent Set

Problem: Independent Set

Instance: A graph G, integer k.
Question: Is there an independent set in G of size k?

Lemma 23.1.

Independent set is in **NP**.

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3SAT $\leq_P$ Independent Set

The reduction 3SAT $\leq_P$ Independent Set

Input: Given a 3CNF formula φ

Goal: Construct a graph G_φ and number k such that G_φ has an independent set of size k if and only if φ is satisfiable.

G_φ should be constructable in time polynomial in size of φ

Importance of reduction: Although 3SAT is much more expressive, it can be reduced to a seemingly specialized Independent Set problem.

Notice: We handle only 3CNF formulas – reduction would not work for other kinds of boolean formulas.

3SAT $\leq_P$ Independent Set

The reduction 3SAT $\leq_P$ Independent Set

Input: Given a $3\mathrm{CNF}$ formula φ

Goal: Construct a graph G_φ and number k such that G_φ has an independent set of size k if and only if φ is satisfiable.

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Interpreting 3SAT

There are two ways to think about **3SAT**

1. Find a way to assign 0/1 (false/true) to the variables such that the formula evaluates to true, that is each clause evaluates to true.
2. Pick a literal from each clause and find a truth assignment to make all of them true. You will fail if two of the literals you pick are in conflict, i.e., you pick x_i and $\neg x_i$

We will take the second view of **3SAT** to construct the reduction.

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The Reduction

1. G_φ will have one vertex for each literal in a clause
2. Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
3. Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict
4. Take k to be the number of clauses

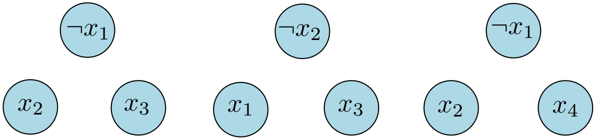

Figure: Graph for $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

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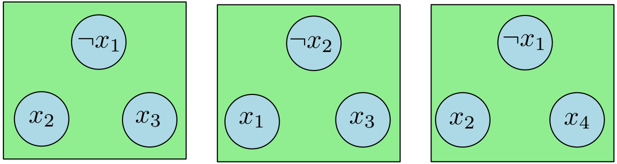

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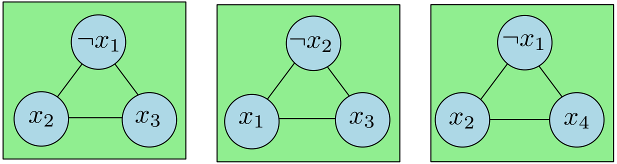

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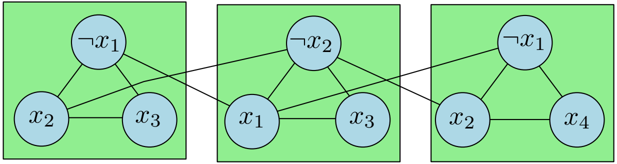

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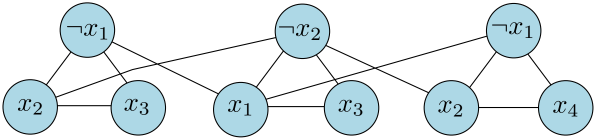

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Correctness

Proposition 23.2.

φ is satisfiable iff $\mathbf{G}_\varphi$ has an independent set of size $\mathbf{k}$ ($=$ number of clauses in φ).

Proof.

$\Rightarrow$ Let $\mathbf{a}$ be the truth assignment satisfying φ

- Pick one of the vertices, corresponding to true literals under $\mathbf{a}$, from each triangle. This is an independent set of the appropriate size. Why?

□

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Correctness

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φ is satisfiable iff $\mathbf{G}_\varphi$ has an independent set of size $\mathbf{k}$ ($=$ number of clauses in φ).

Proof.

$\Leftarrow$ Let $\mathbf{S}$ be an independent set of size $\mathbf{k}$

1. $\mathbf{S}$ must contain exactly one vertex from each clause
2. $\mathbf{S}$ cannot contain vertices labeled by conflicting literals
3. Thus, it is possible to obtain a truth assignment that makes in the literals in $\mathbf{S}$ true; such an assignment satisfies one literal in every clause □

Summary

Theorem 23.3.

Independent set *is* **NP-Complete** *(i.e.,* **NPC***).*

THE END

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(for now)

23.3
NP-Completeness of Hamiltonian Cycle

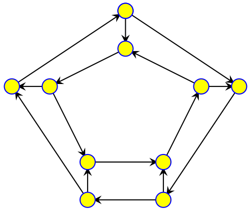
23.3.1
Reduction from 3SAT to Hamiltonian Cycle: Basic idea

Directed Hamiltonian Cycle

Input Given a directed graph $G = (V, E)$ with n vertices

Goal Does G have a Hamiltonian cycle?

- A Hamiltonian cycle is a cycle in the graph that visits every vertex in G exactly once



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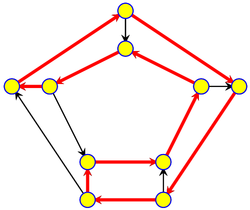
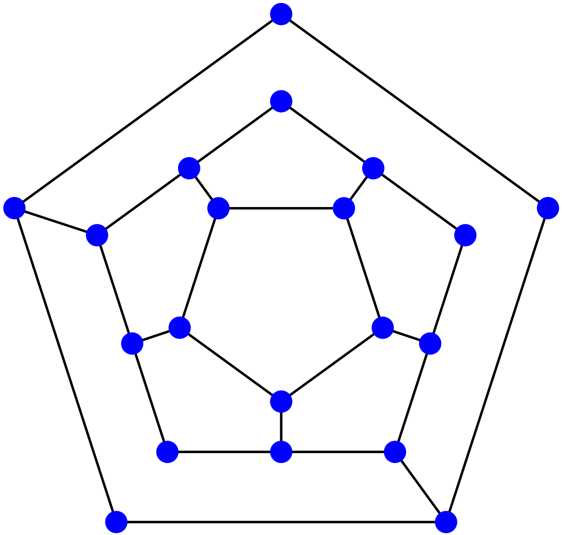


Is the following graph Hamiltonian?



(A) Yes.

(B) No.

Directed Hamiltonian Cycle is **NP-Complete**

- Directed Hamiltonian Cycle is in ***NP***: exercise
- Hardness: We will show **3SAT** $\leq_P$ **Directed Hamiltonian Cycle** .

Reduction construction

From 3SAT to Hamiltonian cycle in directed graph

1. To show reduction, we next describe an algorithm:
   - Input: **3SAT** formula φ
   - Output: A graph G_φ.
   - Running time is polynomial.
   - Requirement: φ is satisfiable $\iff$ G_φ is Hamiltonian.
2. Given **3SAT** formula φ create a graph G_φ such that
   - G_φ has a Hamiltonian cycle if and only if φ is satisfiable
   - G_φ should be constructible from φ by a polynomial time algorithm $\mathcal{A}$
3. Notation: φ has n variables $x_1, x_2, \ldots, x_n$ and m clauses $C_1, C_2, \ldots, C_m$.

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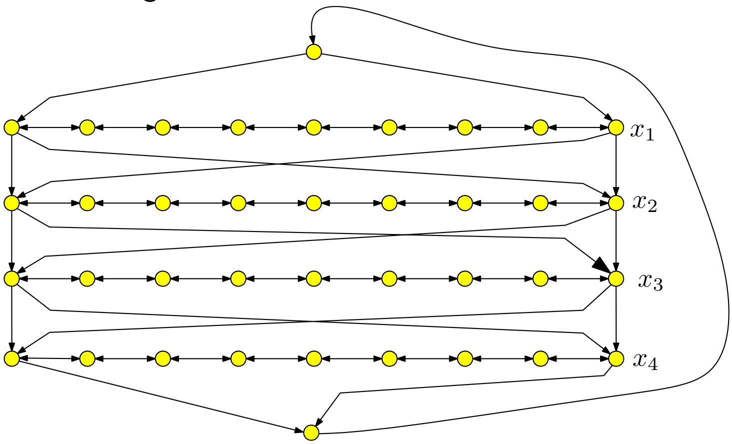
Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with $\mathbf{2^n}$ different Hamiltonian paths, that can encode their assignments.



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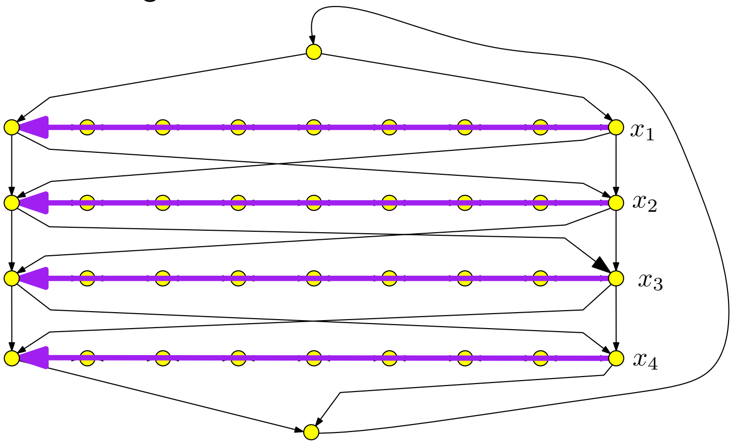

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$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$$

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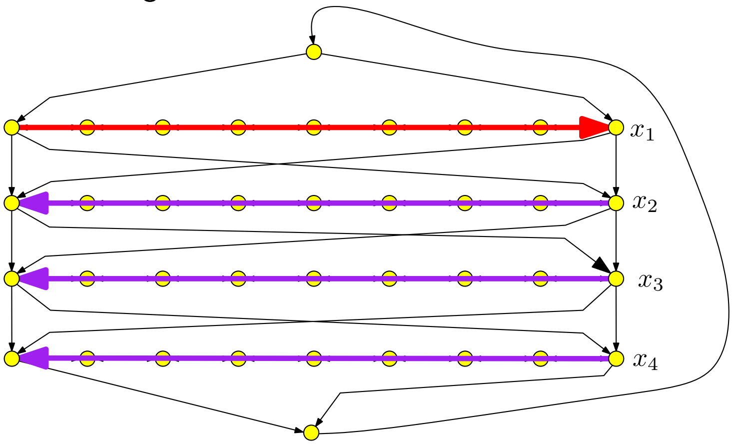

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$$x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 0$$

Encoding assignments

Converting φ to a graph

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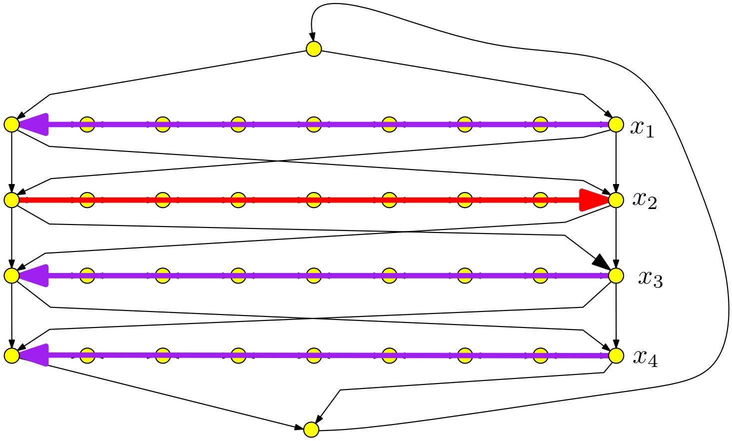

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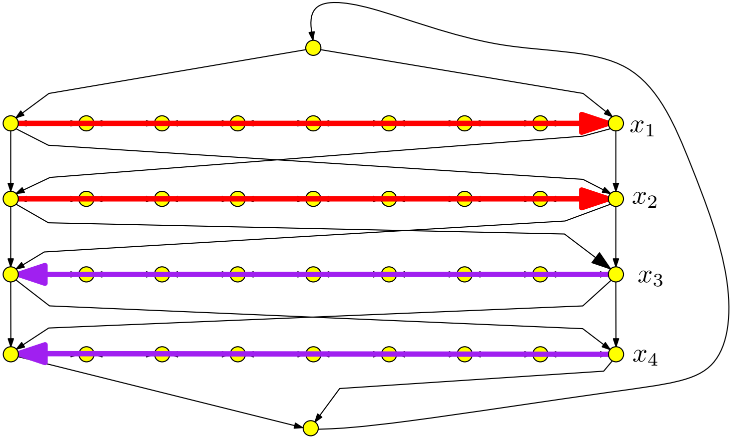
$$x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 0$$

Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with 2^n different Hamiltonian paths, that can encode their assignments.



5

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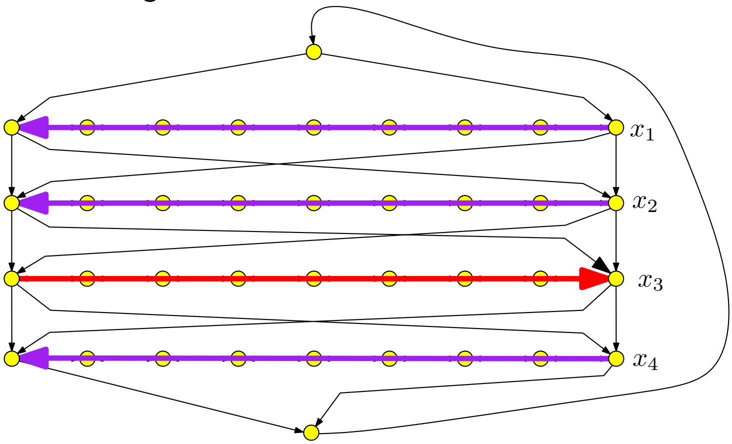
$$x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0$$

Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with 2^n different Hamiltonian paths, that can encode their assignments.



6

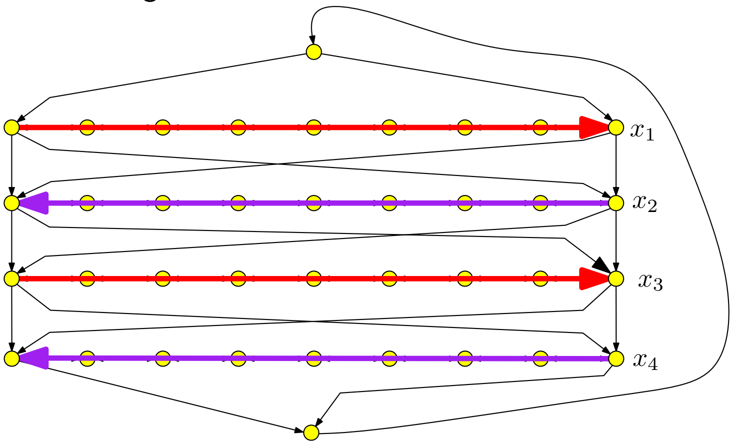
5

$$x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 0$$

Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with 2^n different Hamiltonian paths, that can encode their assignments.



7

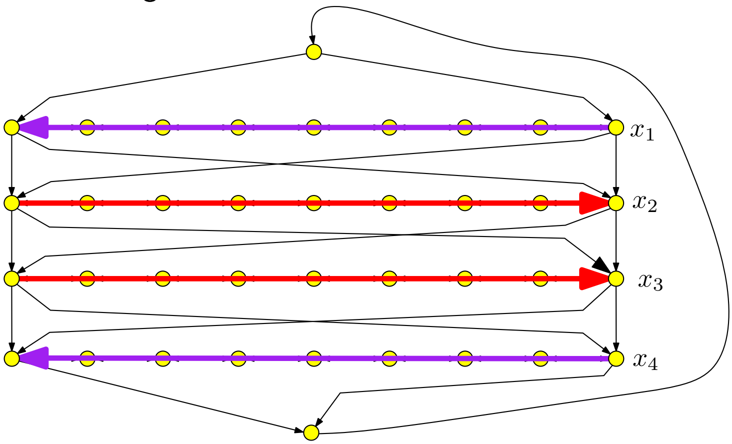
5

$$x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$$

Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with 2^n different Hamiltonian paths, that can encode their assignments.



8

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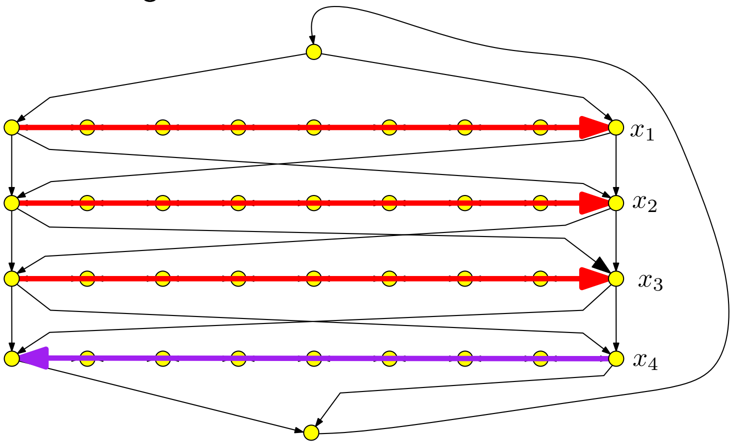
$$x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 0$$

Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with 2^n different Hamiltonian paths, that can encode their assignments.



9

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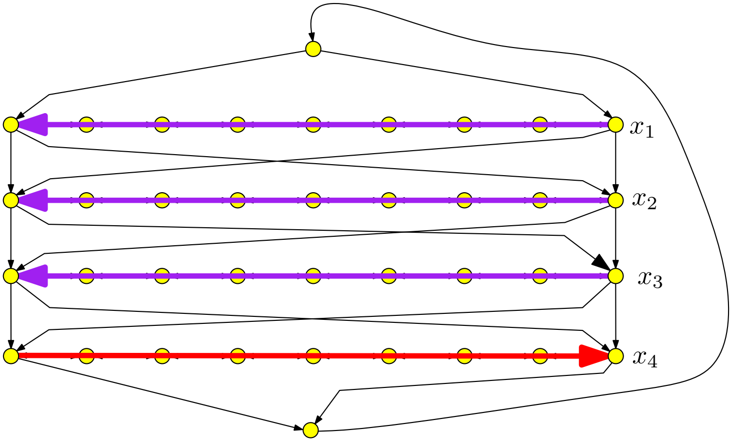
$$x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 01$$

Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with 2^n different Hamiltonian paths, that can encode their assignments.



10

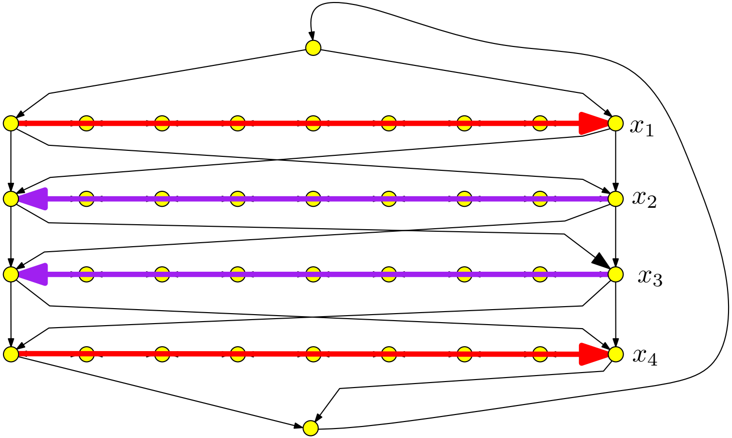
5

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 1$$

Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with 2^n different Hamiltonian paths, that can encode their assignments.



11

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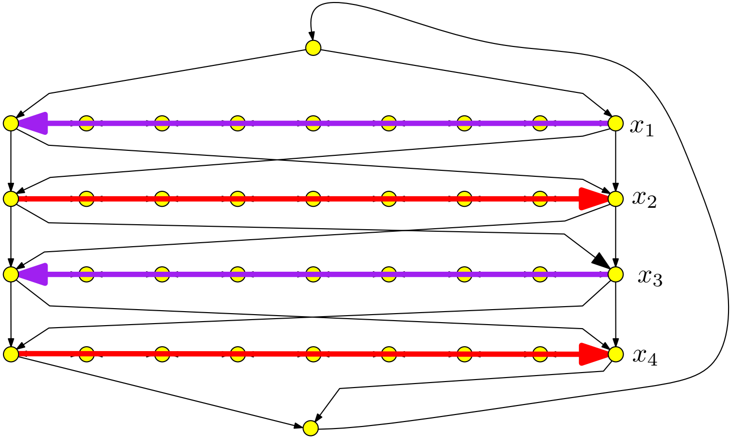
$$x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 1$$

Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with 2^n different Hamiltonian paths, that can encode their assignments.



12

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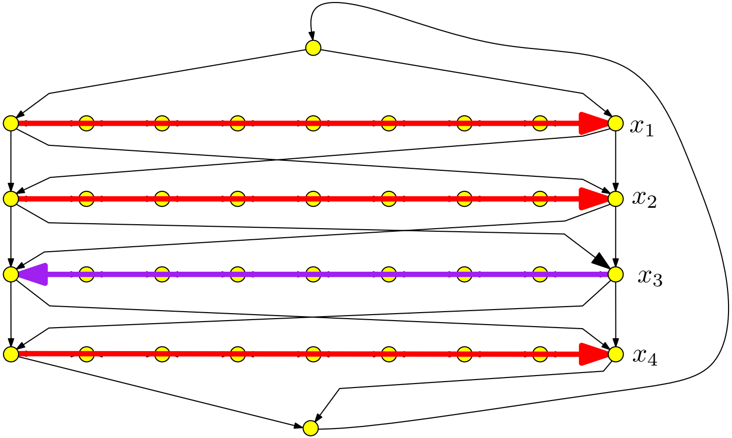
$x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 1$

Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with 2^n different Hamiltonian paths, that can encode their assignments.



13

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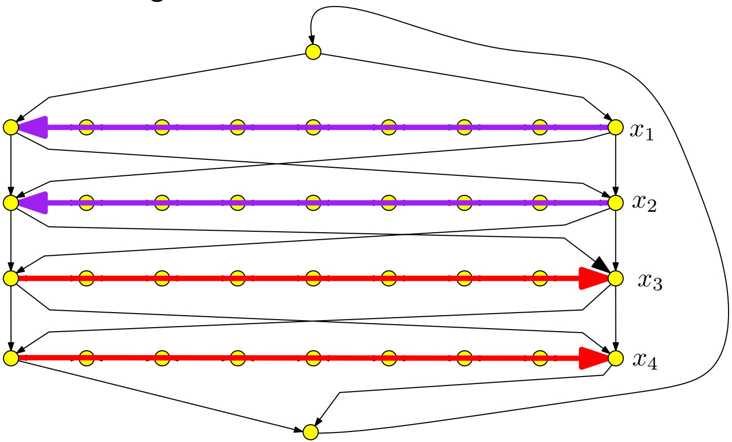
$$x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1$$

Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with 2^n different Hamiltonian paths, that can encode their assignments.



14

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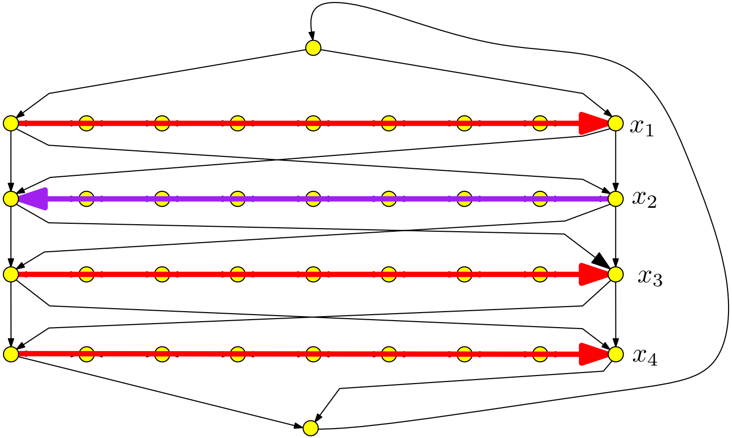
$$x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1$$

Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with 2^n different Hamiltonian paths, that can encode their assignments.



15

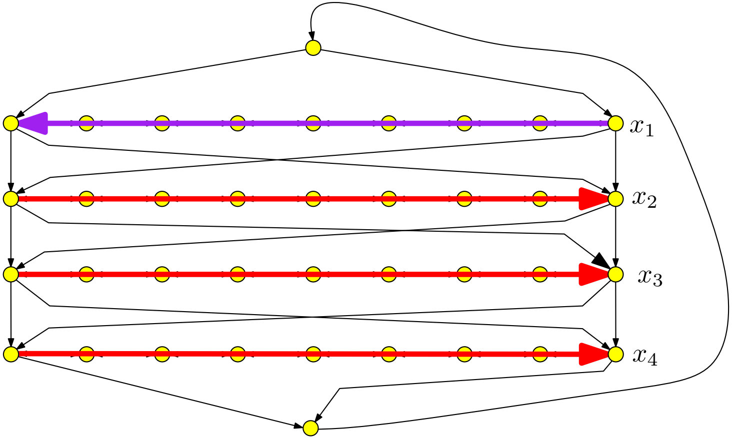
5

$$x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 1$$

Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with 2^n different Hamiltonian paths, that can encode their assignments.



16

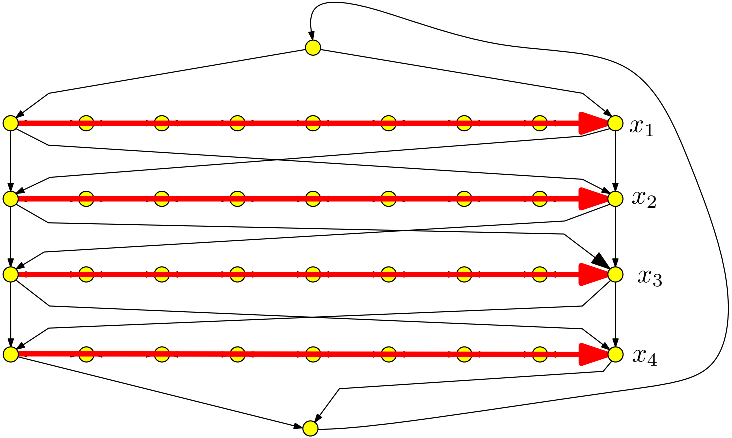
5

$$x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 1$$

Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with 2^n different Hamiltonian paths, that can encode their assignments.



17

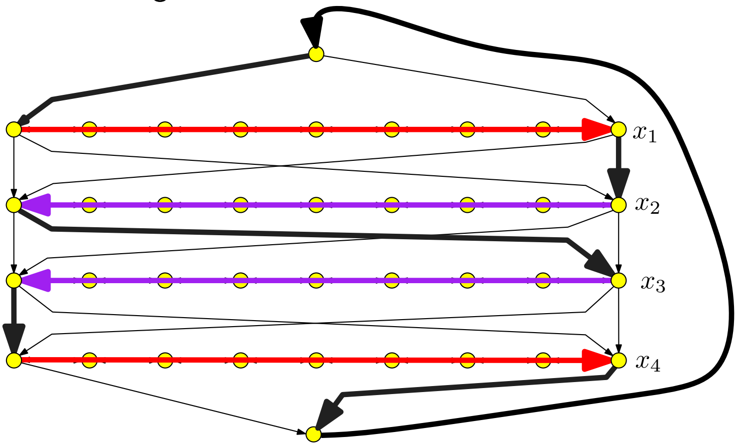
5

$$x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1$$

Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with 2^n different Hamiltonian paths, that can encode their assignments.



18

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THE END

...

(for now)

23.3.2
The reduction: Encoding the formula constraints

3SAT $\leq_P$ Directed Hamiltonian Cycle

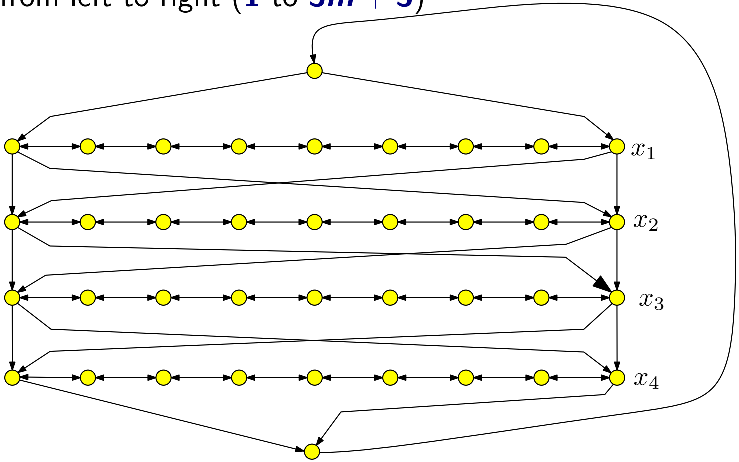
Input: φ formula.
Output: Graph G_φ.

Saw: How to encode assignments...
Now need to encode constraints of φ.

The reduction algorithm: Phase I

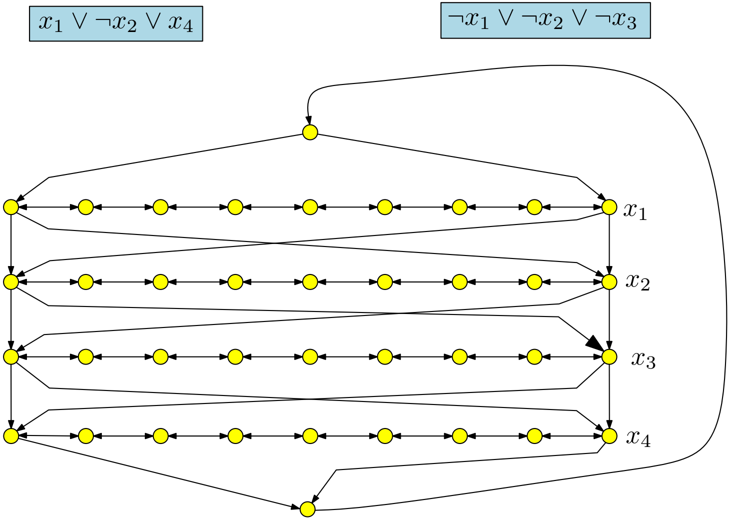
Converting φ to a graph

- Traverse path $\boldsymbol{i}$ from left to right iff $\boldsymbol{x_i}$ is set to true
- Each path has $\mathbf{3(m+1)}$ nodes where $\boldsymbol{m}$ is number of clauses in φ; nodes numbered from left to right ($\mathbf{1}$ to $\mathbf{3m+3}$)



The Reduction algorithm: Phase II

- Add vertex c_j for clause C_j. c_j has edge from vertex $3j$ and to vertex $3j + 1$ on path i if x_i appears in clause C_j, and has edge from vertex $3j + 1$ and to vertex $3j$ if $\neg x_i$ appears in C_j.



The Reduction algorithm: Phase II

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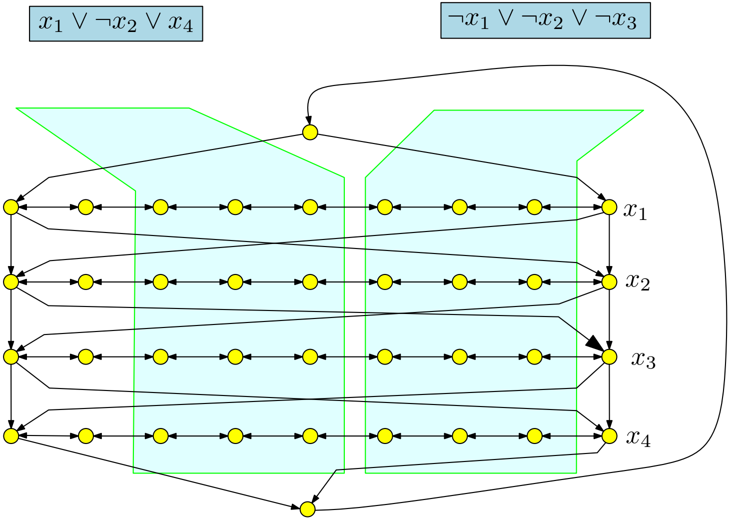
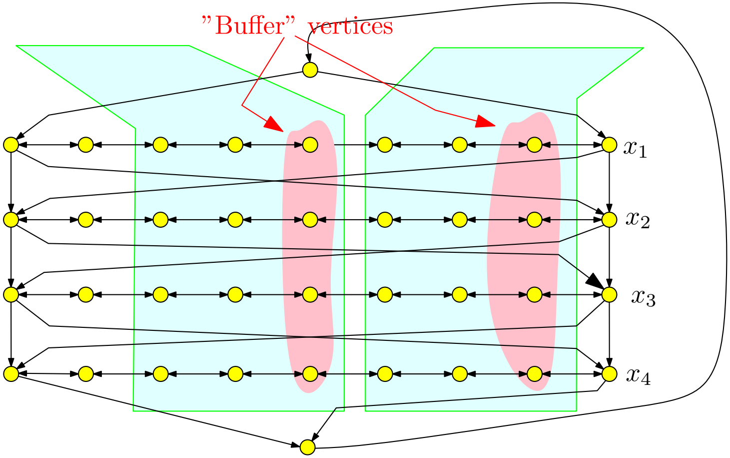


The Reduction algorithm: Phase II

- Add vertex c_j for clause C_j. c_j has edge from vertex $3j$ and to vertex $3j+1$ on path i if x_i appears in clause C_j, and has edge from vertex $3j+1$ and to vertex $3j$ if $\neg x_i$ appears in C_j.

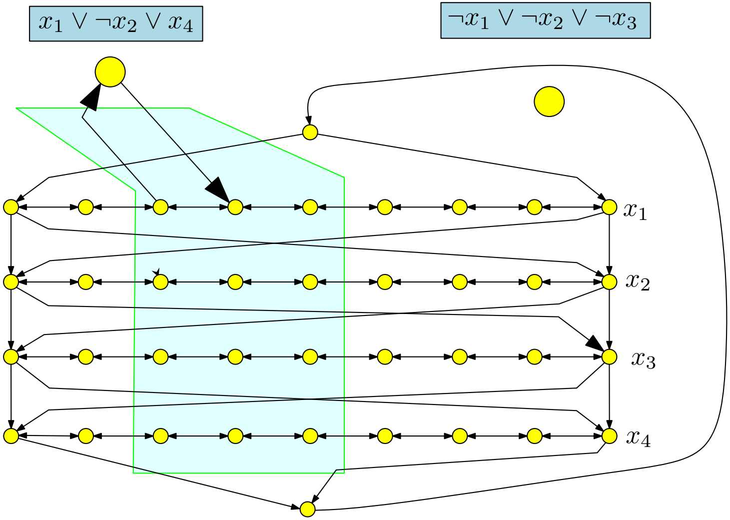
$x_1 \vee \neg x_2 \vee x_4$

$\neg x_1 \vee \neg x_2 \vee \neg x_3$



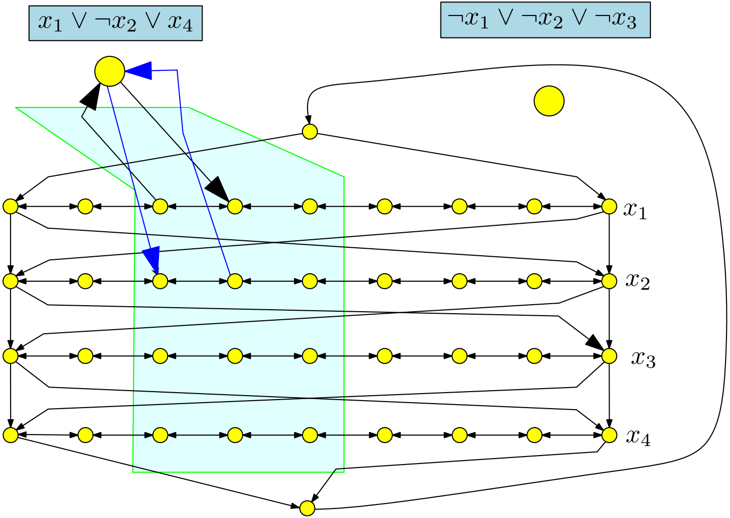
The Reduction algorithm: Phase II

- Add vertex c_j for clause C_j. c_j has edge from vertex $3j$ and to vertex $3j + 1$ on path i if x_i appears in clause C_j, and has edge from vertex $3j + 1$ and to vertex $3j$ if $\neg x_i$ appears in C_j.



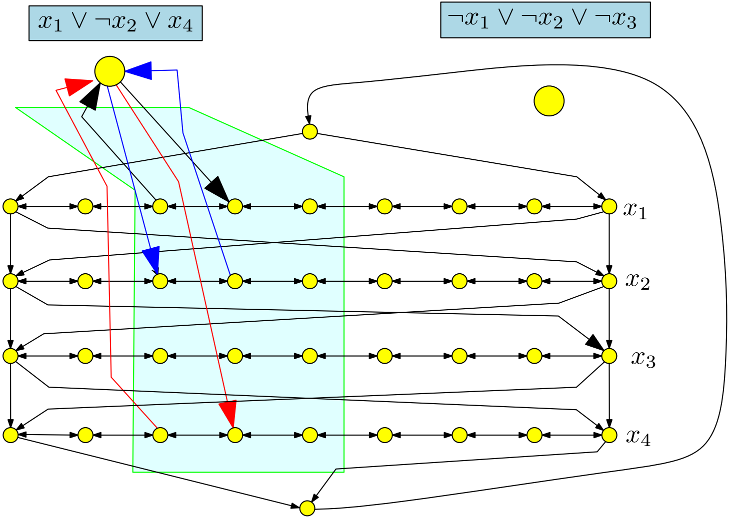
The Reduction algorithm: Phase II

- Add vertex c_j for clause C_j. c_j has edge from vertex $3j$ and to vertex $3j + 1$ on path i if x_i appears in clause C_j, and has edge from vertex $3j + 1$ and to vertex $3j$ if $\neg x_i$ appears in C_j.



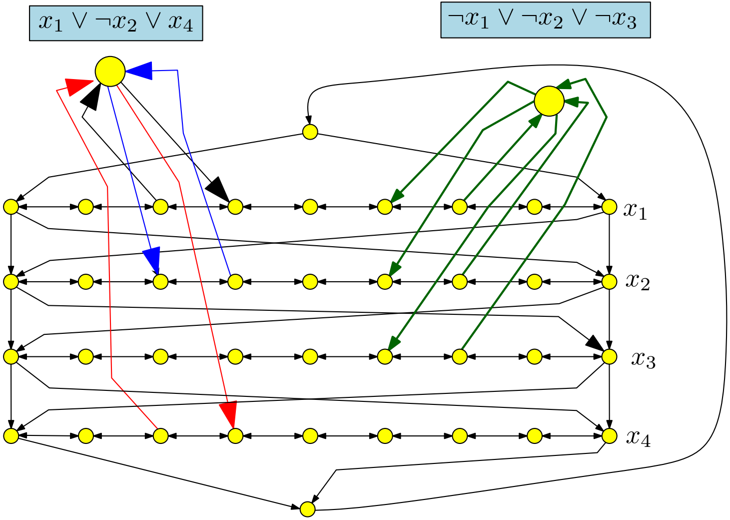
The Reduction algorithm: Phase II

- Add vertex c_j for clause C_j. c_j has edge from vertex $3j$ and to vertex $3j + 1$ on path i if x_i appears in clause C_j, and has edge from vertex $3j + 1$ and to vertex $3j$ if $\neg x_i$ appears in C_j.



The Reduction algorithm: Phase II

- Add vertex c_j for clause C_j. c_j has edge from vertex $3j$ and to vertex $3j + 1$ on path i if x_i appears in clause C_j, and has edge from vertex $3j + 1$ and to vertex $3j$ if $\neg x_i$ appears in C_j.



THE END

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(for now)

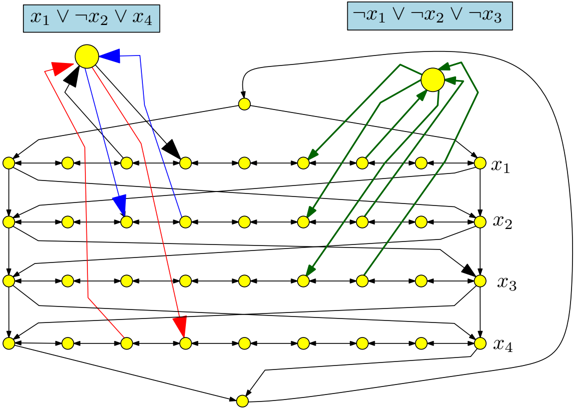
23.3.3

If there is a satisfying assignment, then there is a Hamiltonian cycle

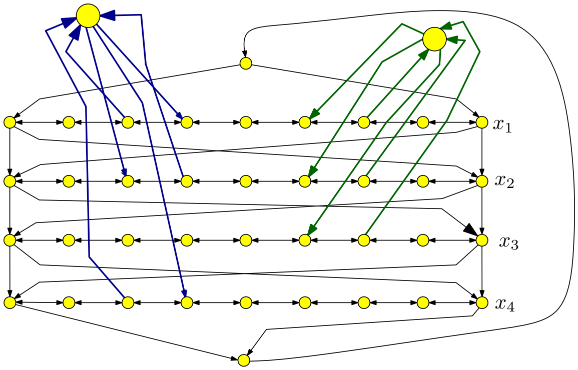
From satisfying assignment to Hamiltonian cycle: By figure

3SAT formula φ:

$$\varphi = \left(x_1 \vee \neg x_2 \vee x_4\right) \wedge \left(\neg x_1 \vee \neg x_2 \vee \neg x_3\right)$$



From satisfying assignment to Hamiltonian cycle: By figure



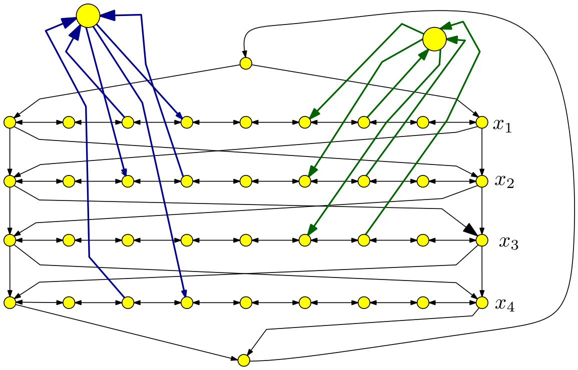
3SAT formula φ:

$$\varphi = \left(x_1 \vee \neg x_2 \vee x_4 \right) \wedge \left(\neg x_1 \vee \neg x_2 \vee \neg x_3 \right)$$

From satisfying assignment to Hamiltonian cycle: By figure

3SAT formula φ:

$$\varphi = \left(x_1 \vee \neg x_2 \vee x_4\right) \wedge \left(\neg x_1 \vee \neg x_2 \vee \neg x_3\right)$$



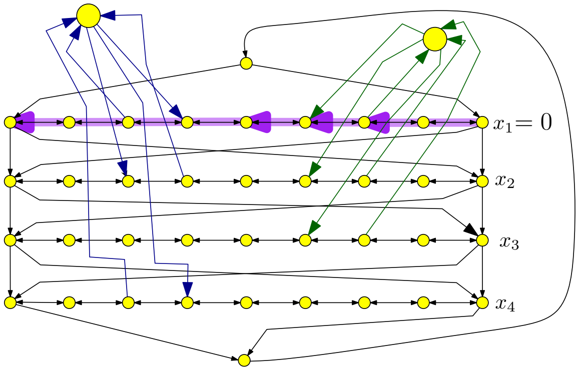
A satisfying assignment:
$x_1 = \mathbf{0}$, $x_2 = \mathbf{1}$, $x_3 = \mathbf{0}$, $x_4 = \mathbf{1}$

From satisfying assignment to Hamiltonian cycle: By figure

3SAT formula φ:

$$\varphi = \left(x_1 \vee \neg x_2 \vee x_4\right) \wedge \left(\neg x_1 \vee \neg x_2 \vee \neg x_3\right)$$

A satisfying assignment:
$x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$

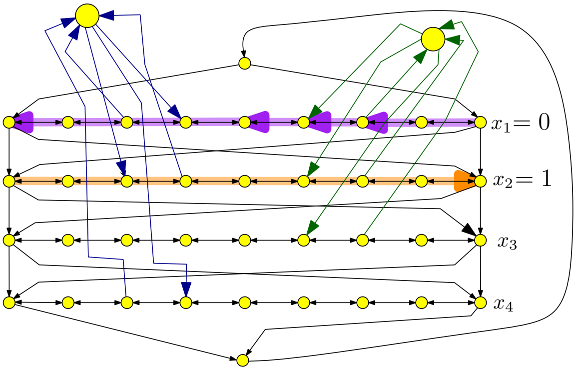


From satisfying assignment to Hamiltonian cycle: By figure

3SAT formula φ:

$$\varphi = \left(x_1 \vee \neg x_2 \vee x_4\right) \wedge \left(\neg x_1 \vee \neg x_2 \vee \neg x_3\right)$$

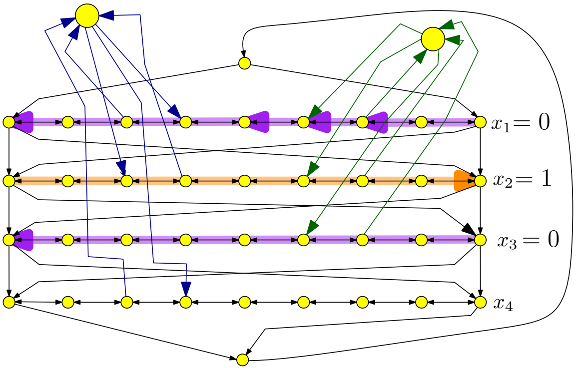
A satisfying assignment:
$x_1 = \mathbf{0}$, $x_2 = \mathbf{1}$, $x_3 = \mathbf{0}$, $x_4 = \mathbf{1}$



From satisfying assignment to Hamiltonian cycle: By figure

3SAT formula φ:

$$\varphi = \left(x_1 \vee \neg x_2 \vee x_4\right) \wedge \left(\neg x_1 \vee \neg x_2 \vee \neg x_3\right)$$



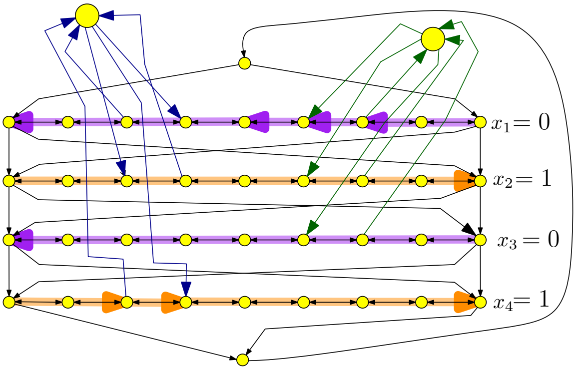
A satisfying assignment:
$x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$

From satisfying assignment to Hamiltonian cycle: By figure

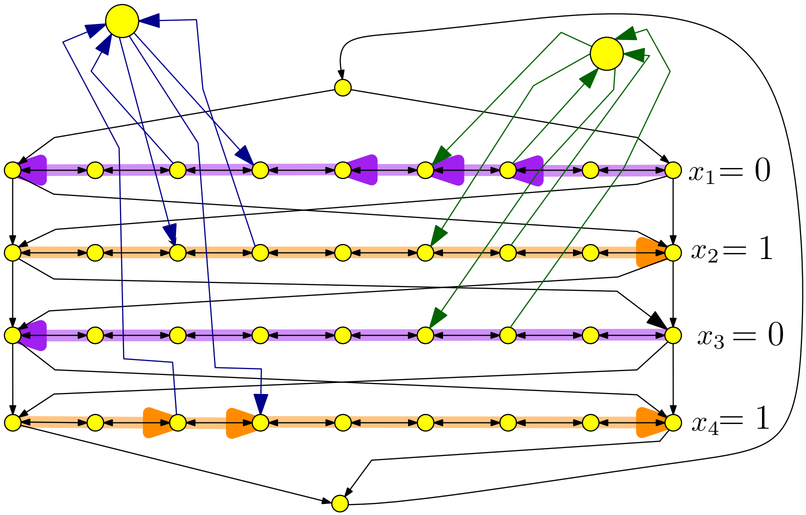
3SAT formula φ:

$$\varphi = \left(x_1 \vee \neg x_2 \vee x_4\right) \wedge \left(\neg x_1 \vee \neg x_2 \vee \neg x_3\right)$$

A satisfying assignment:
$x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$

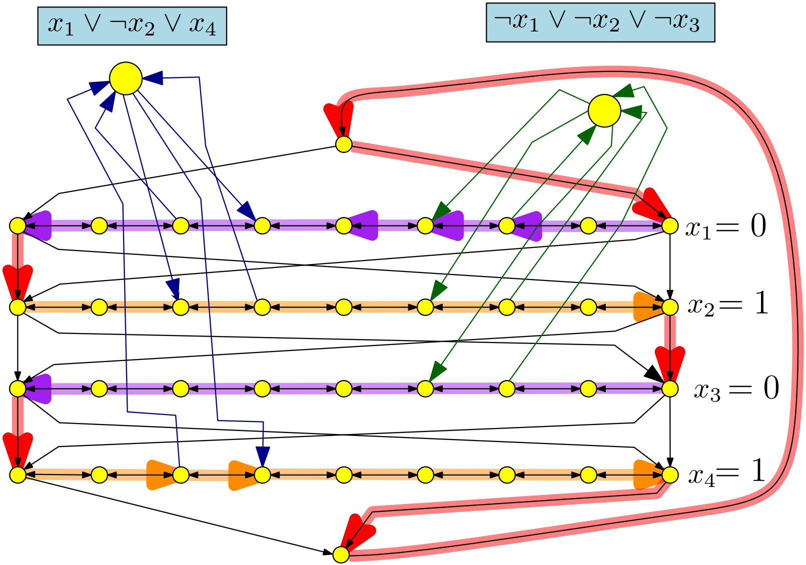


Reduction: Satisfying assignment ⇒ Hamiltonian cycle



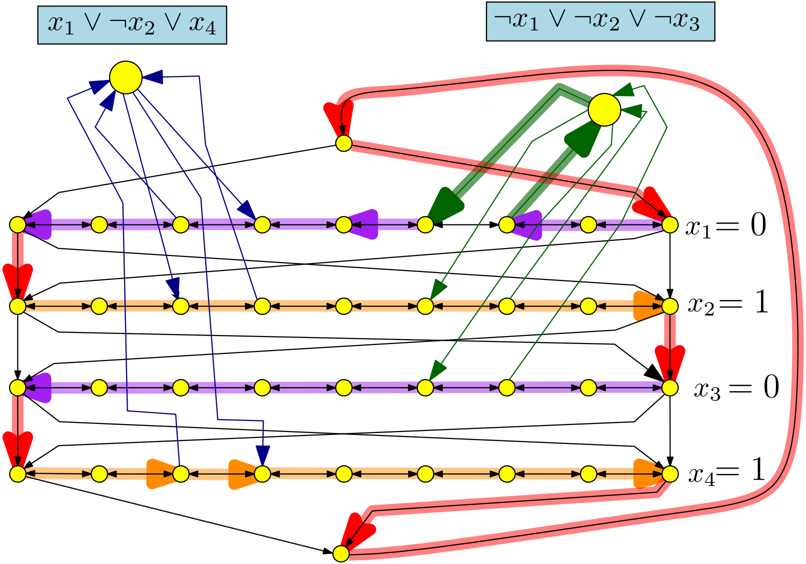
Satisfying assignment: $\boldsymbol{x_1 = 0}$, $\boldsymbol{x_2 = 1}$, $\boldsymbol{x_3 = 0}$, $\boldsymbol{x_4 = 1}$

Reduction: Satisfying assignment ⇒ Hamiltonian cycle



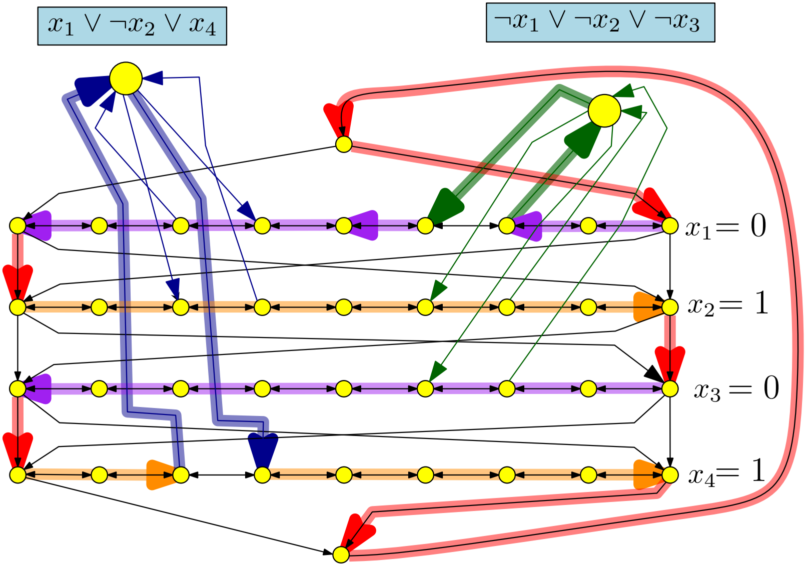
Satisfying assignment: $\boldsymbol{x_1 = 0}$, $\boldsymbol{x_2 = 1}$, $\boldsymbol{x_3 = 0}$, $\boldsymbol{x_4 = 1}$

Reduction: Satisfying assignment ⇒ Hamiltonian cycle



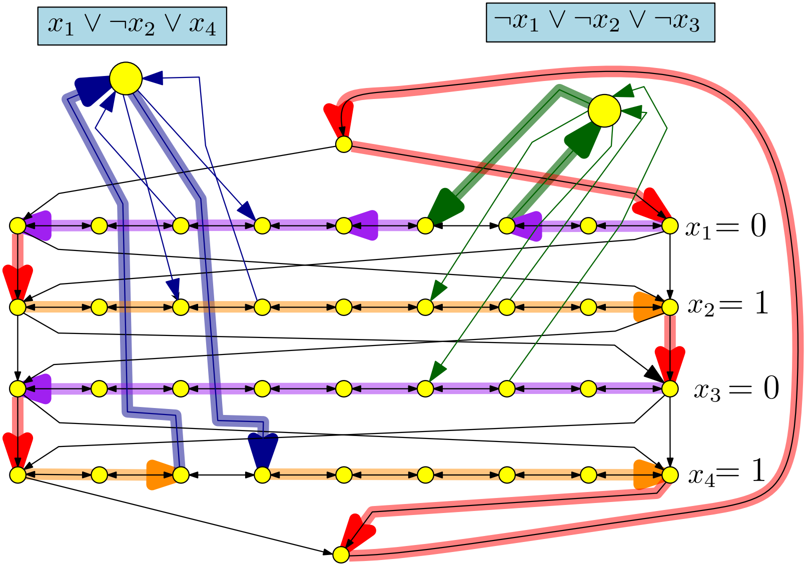
Satisfying assignment: $\boldsymbol{x_1 = 0}$, $\boldsymbol{x_2 = 1}$, $\boldsymbol{x_3 = 0}$, $\boldsymbol{x_4 = 1}$

Reduction: Satisfying assignment ⇒ Hamiltonian cycle



Satisfying assignment: $\boldsymbol{x_1 = 0}$, $\boldsymbol{x_2 = 1}$, $\boldsymbol{x_3 = 0}$, $\boldsymbol{x_4 = 1}$

Reduction: Satisfying assignment ⇒ Hamiltonian cycle



Satisfying assignment: $\boldsymbol{x_1 = 0}$, $\boldsymbol{x_2 = 1}$, $\boldsymbol{x_3 = 0}$, $\boldsymbol{x_4 = 1}$

Conclude: If φ has a satisfying assignment then there is an Hamiltonian cycle in G_φ.

Correctness Proof

Lemma 23.1.

φ has a satisfying assignment $\alpha \implies G_\varphi$ has a Hamiltonian cycle.

Proof.

Let a be the satisfying assignment for φ. Define Hamiltonian cycle as follows

- If $\alpha(x_i) = 1$ then traverse path i from left to right
- If $\alpha(x_i) = 0$ then traverse path i from right to left
- For each clause, path of at least one variable is in the "right" direction to splice in the node corresponding to clause
- Clearly, resulting cycle is Hamiltonian. □

THE END

...

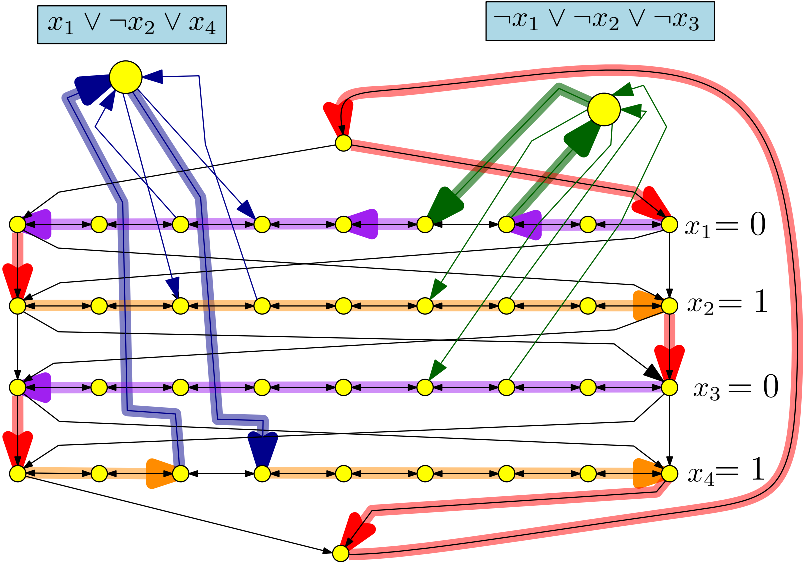
(for now)

23.3.4

If there is a Hamiltonian cycle $\implies$ $\exists$satisfying assignment

Reduction: Hamiltonian cycle $\implies \exists$ satisfying assignment

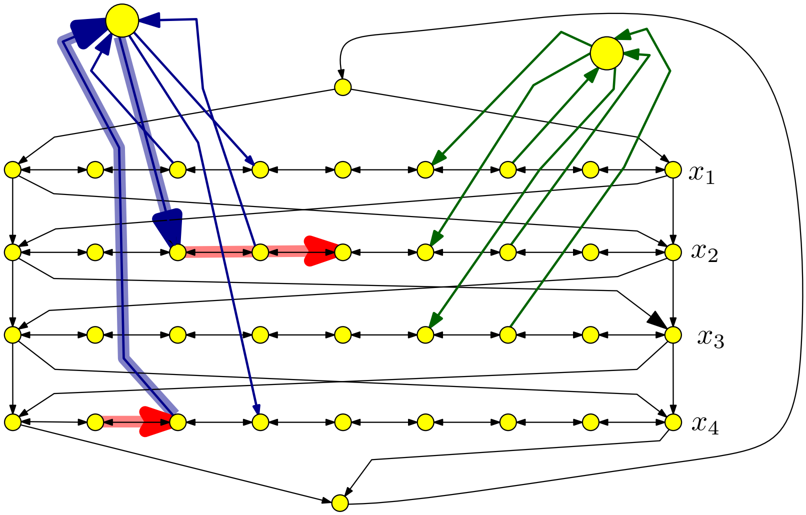
We are given a Hamiltonian cycle in $\boldsymbol{G_\varphi}$:



Want to extract satisfying assignment...

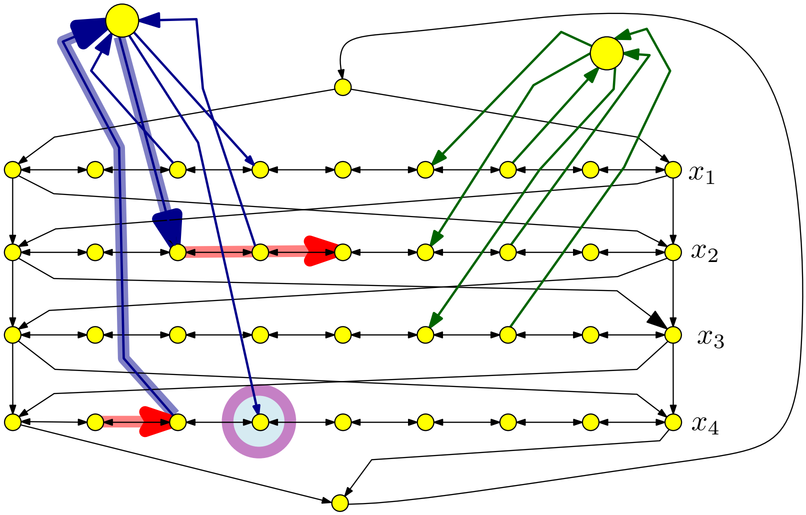
Reduction: Hamiltonian cycle $\implies \exists$ satisfying assignment

No shenanigan: Hamiltonian cycle can not leave a row in the middle



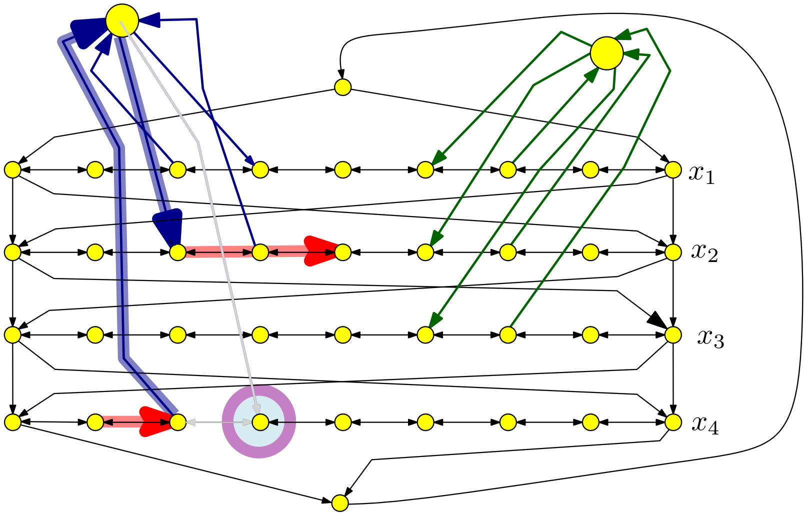
Reduction: Hamiltonian cycle $\Longrightarrow$ $\exists$ satisfying assignment

No shenanigan: Hamiltonian cycle can not leave a row in the middle



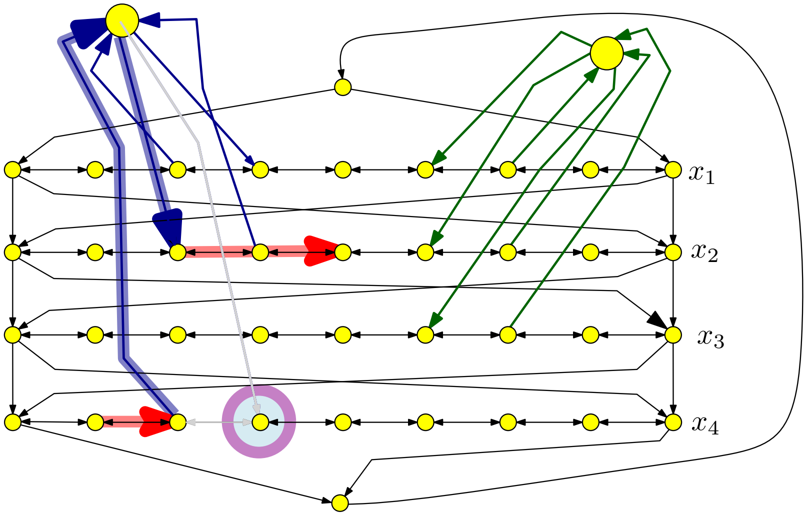
Reduction: Hamiltonian cycle $\implies$ $\exists$ satisfying assignment

No shenanigan: Hamiltonian cycle can not leave a row in the middle



Reduction: Hamiltonian cycle $\implies$ $\exists$ satisfying assignment

No shenanigan: Hamiltonian cycle can not leave a row in the middle



Conclude: Hamiltonian cycle must go through each row completely from left to right, or right to left. As such, can be interpreted as a valid assignment.

Hamiltonian Cycle $\Rightarrow$ Satisfying assignment

Suppose Π is a Hamiltonian cycle in G_φ

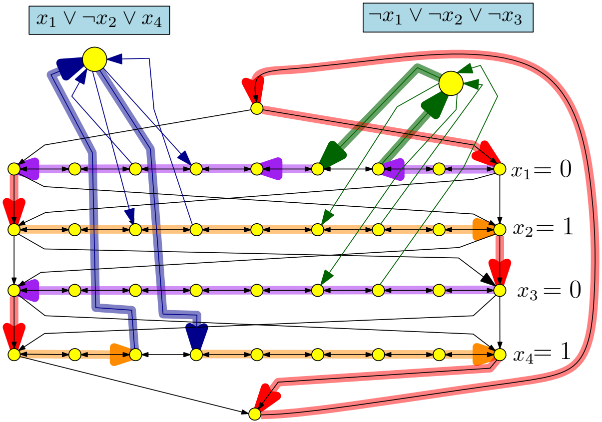
- If Π enters c_j (vertex for clause C_j) from vertex $3j$ on path i then it must leave the clause vertex on edge to $3j+1$ on the <u>same path i</u>
  - If not, then only unvisited neighbor of $3j+1$ on path i is $3j+2$
  - Thus, we don't have two unvisited neighbors (one to enter from, and the other to leave) to have a Hamiltonian Cycle
- Similarly, if Π enters c_j from vertex $3j+1$ on path i then it must leave the clause vertex c_j on edge to $3j$ on path i

Hamiltonian Cycle $\implies$ Satisfying assignment (contd)

- Thus, vertices visited immediately before and after C_i are connected by an edge
- We can remove c_j from cycle, and get Hamiltonian cycle in $G - c_j$
- Consider Hamiltonian cycle in $G - \{c_1, \ldots c_m\}$; it traverses each path in only one direction, which determines the truth assignment

Hamiltonian Cycle $\implies$ Satisfying assignment (contd)

- Thus, vertices visited immediately before and after C_i are connected by an edge
- We can remove c_j from cycle, and get Hamiltonian cycle in $G - c_j$
- Consider Hamiltonian cycle in $G - \{c_1, \ldots c_m\}$; it traverses each path in only one direction, which determines the truth assignment



Hamiltonian Cycle $\implies$ Satisfying assignment (contd)

- Thus, vertices visited immediately before and after C_i are connected by an edge
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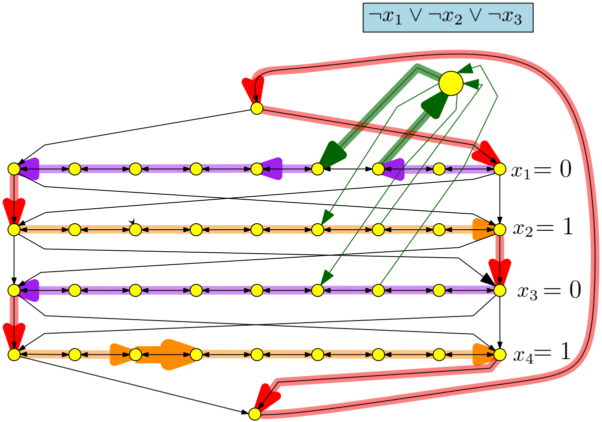

Hamiltonian Cycle $\implies$ Satisfying assignment (contd)

- Thus, vertices visited immediately before and after C_i are connected by an edge
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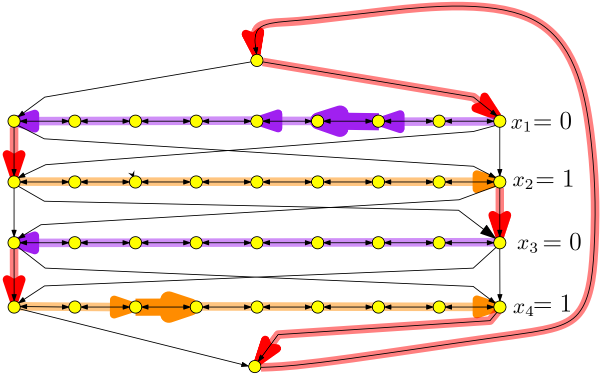

Correctness Proof

We just proved:

Lemma 23.2.

G_φ has a Hamiltonian cycle $\implies$ φ has a satisfying assignment α.

Lemma 23.3.

φ has a satisfying assignment iff G_φ has a Hamiltonian cycle.

Proof.

Follows from **Lemma 23.1** and **Lemma 23.2**. □

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What we did:

1. Showed that **Directed Hamiltonian Cycle** is in **NP**.
2. Provided a polynomial time reduction from **3SAT** to **Directed Hamiltonian Cycle**.
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*The problem **Hamiltonian Cycle** in directed graphs is **NP-Complete**.*

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Theorem 23.4.

*The problem **Hamiltonian Cycle** in directed graphs is **NP-Complete**.*

THE END

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(for now)

23.4
Hamiltonian cycle in undirected graph

Hamiltonian Cycle

Problem 23.1.

Input *Given undirected graph* $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$

Goal *Does* $\boldsymbol{G}$ *have a Hamiltonian cycle? That is, is there a cycle that visits every vertex exactly one (except start and end vertex)?*

NP-Completeness

Theorem 23.2.

Hamiltonian cycle *problem for* ***undirected*** *graphs is* **NP-Complete**.

Proof.

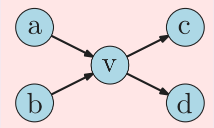
- The problem is in **NP**; proof left as exercise.
- Hardness proved by reducing Directed Hamiltonian Cycle to this problem □

Reduction Sketch

Goal: Given directed graph G, need to construct undirected graph G' such that G has Hamiltonian Path iff G' has Hamiltonian path

Reduction

- Replace each vertex v by 3 vertices: v_{in}, v, and v_{out}
- A directed edge (a, b) is replaced by edge (a_{out}, b_{in})



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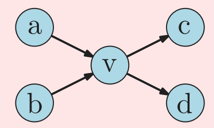

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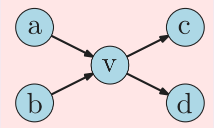

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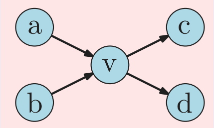
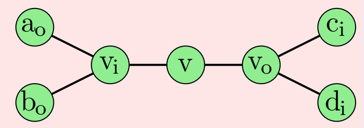
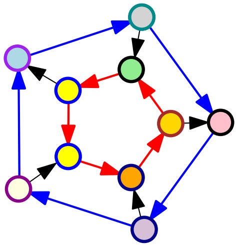


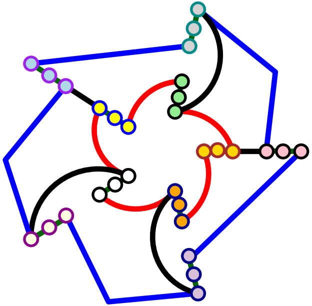


Hamiltonian cycle reduction

Undirected to directed case



$\Rightarrow$



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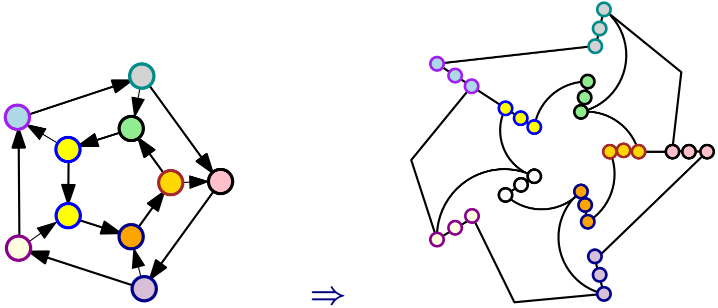

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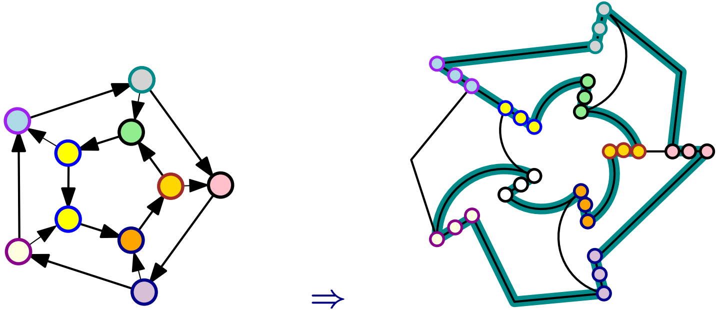

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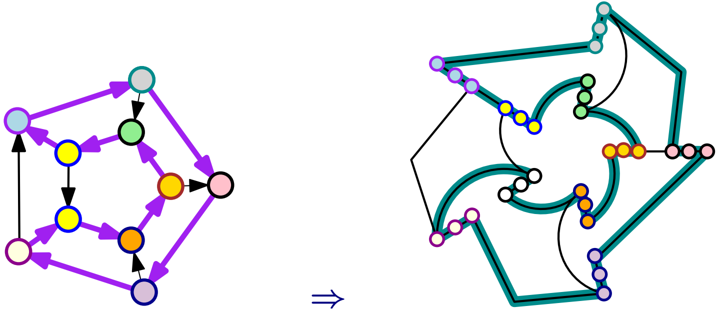

Reduction: Wrap-up

- The reduction is polynomial time (exercise)
- The reduction is correct (exercise)

THE END

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(for now)