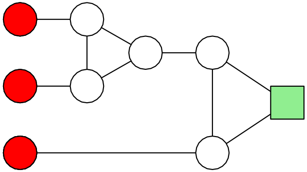
24.3.3.2
The clause gadget

3 color this gadget.

Clicker question

You are given three colors: red, green and blue. Can the following graph be three colored in a valid way (assuming the two nodes are already colored as indicated).



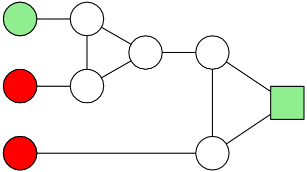
(A) Yes.

(B) No.

3 color this gadget II

Clicker question

You are given three colors: red, green and blue. Can the following graph be three colored in a valid way (assuming the two nodes are already colored as indicated).



(A) Yes.

(B) No.

Clause Satisfiability Gadget

1. For each clause $\boldsymbol{C_j} = (\boldsymbol{a} \lor \boldsymbol{b} \lor \boldsymbol{c})$, create a small gadget graph
   - gadget graph connects to nodes corresponding to $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$
   - needs to implement OR
2. OR-gadget-graph:

Clause Satisfiability Gadget

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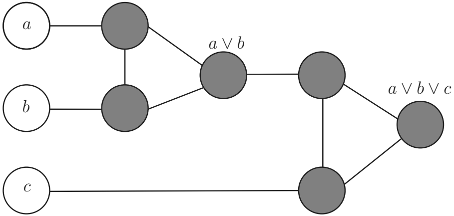


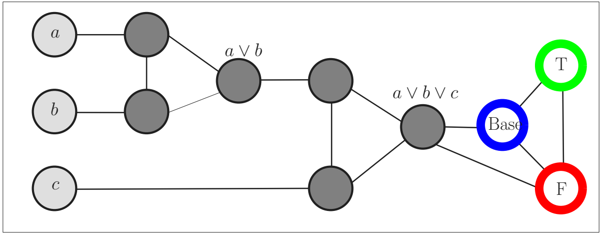
OR-Gadget Graph

Property: if $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are colored False in a 3-coloring then output node of OR-gadget has to be colored False.

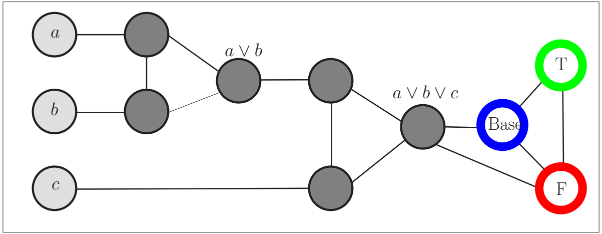
Property: if one of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is colored True then OR-gadget can be 3-colored such that output node of OR-gadget is colored True.

Reduction

- create triangle with nodes True, False, Base
- for each variable $\boldsymbol{x_i}$ two nodes $\boldsymbol{v_i}$ and $\boldsymbol{\bar{v}_i}$ connected in a triangle with common Base
- for each clause $\boldsymbol{C_j} = (\boldsymbol{a} \vee \boldsymbol{b} \vee \boldsymbol{c})$, add OR-gadget graph with input nodes $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and connect output node of gadget to both False and Base



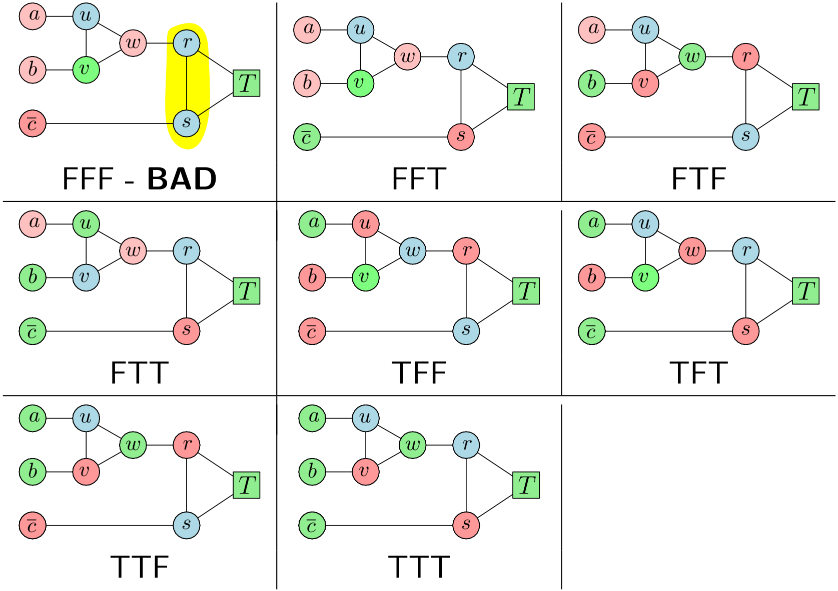
Reduction



Claim 24.1.

No legal **3***-coloring of above graph (with coloring of nodes* ***T***, ***F***, ***B*** *fixed) in which* ***a***, ***b***, ***c*** *are colored False. If any of* ***a***, ***b***, ***c*** *are colored True then there is a legal* **3***-coloring of above graph.*

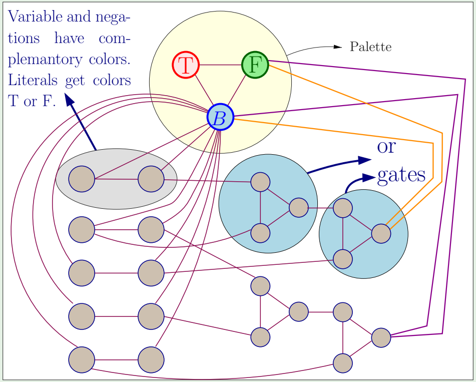
3 coloring of the clause gadget



Reduction Outline

Example 24.2.

$\varphi = (u \vee \neg v \vee w) \wedge (v \vee x \vee \neg y)$



Correctness of Reduction

φ is satisfiable implies G_φ is 3-colorable

- if x_i is assigned True, color v_i True and $\bar{v}_i$ False
- for each clause $C_j = (a \lor b \lor c)$ at least one of a, b, c is colored True. OR-gadget for C_j can be 3-colored such that output is True.

G_φ is 3-colorable implies φ is satisfiable

- if v_i is colored True then set x_i to be True, this is a legal truth assignment
- consider any clause $C_j = (a \lor b \lor c)$. it cannot be that all a, b, c are False. If so, output of OR-gadget for C_j has to be colored False but output is connected to Base and False!

Correctness of Reduction

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- if x_i is assigned True, color v_i True and $\bar{v}_i$ False
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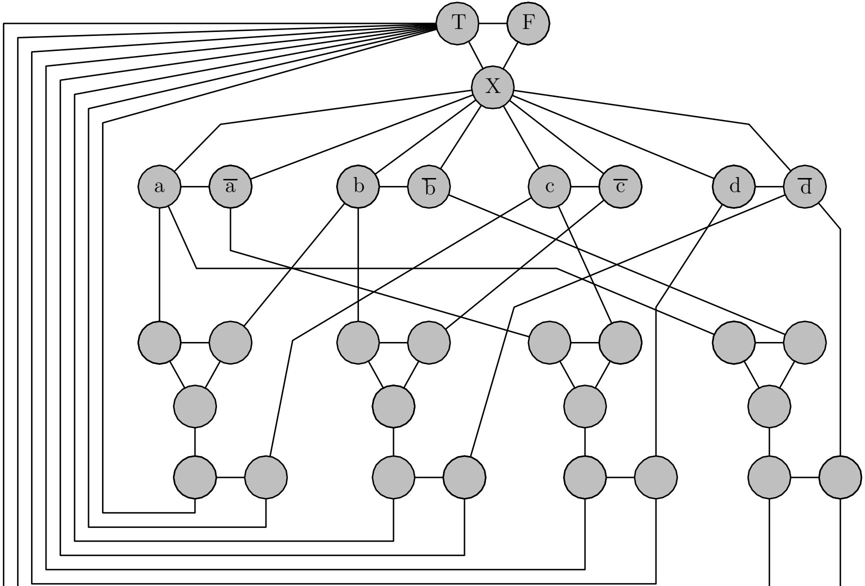
- consider any clause $C_j = (a \vee b \vee c)$. it cannot be that all a, b, c are False. If so, output of OR-gadget for C_j has to be colored False but output is connected to Base and False!

Graph generated in reduction...

... from 3SAT to 3COLOR

$(\boldsymbol{a} \vee \boldsymbol{b} \vee \boldsymbol{c}) \wedge (\boldsymbol{b} \vee \overline{\boldsymbol{c}} \vee \overline{\boldsymbol{d}}) \wedge (\overline{\boldsymbol{a}} \vee \boldsymbol{c} \vee \boldsymbol{d}) \wedge (\boldsymbol{a} \vee \overline{\boldsymbol{b}} \vee \overline{\boldsymbol{d}})$



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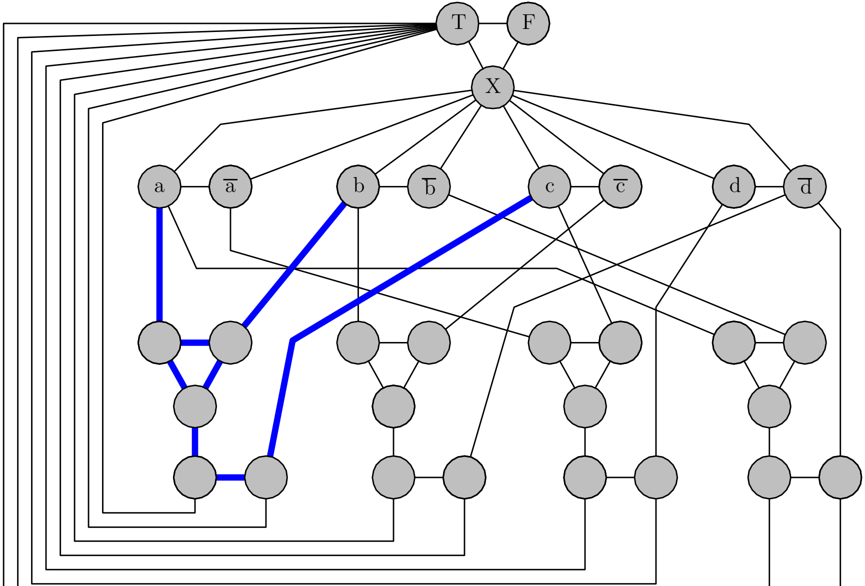
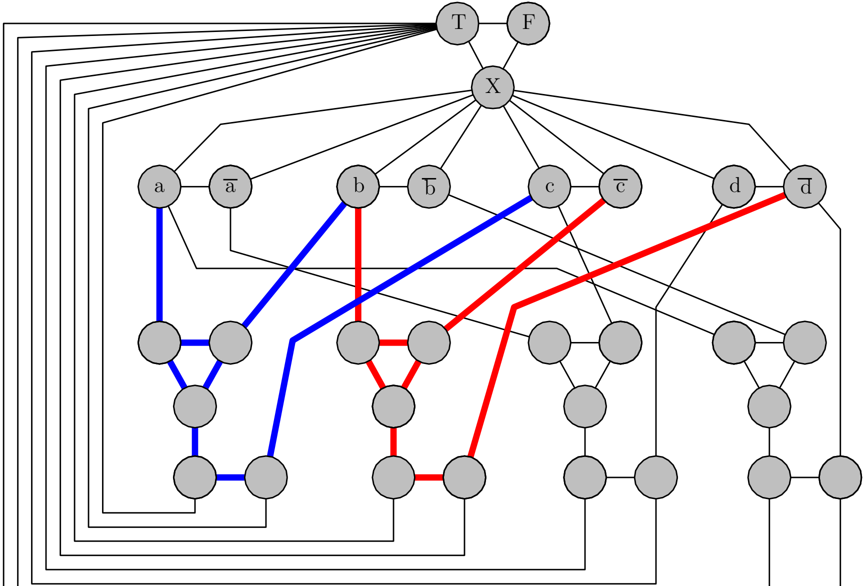


Graph generated in reduction...

... from 3SAT to 3COLOR

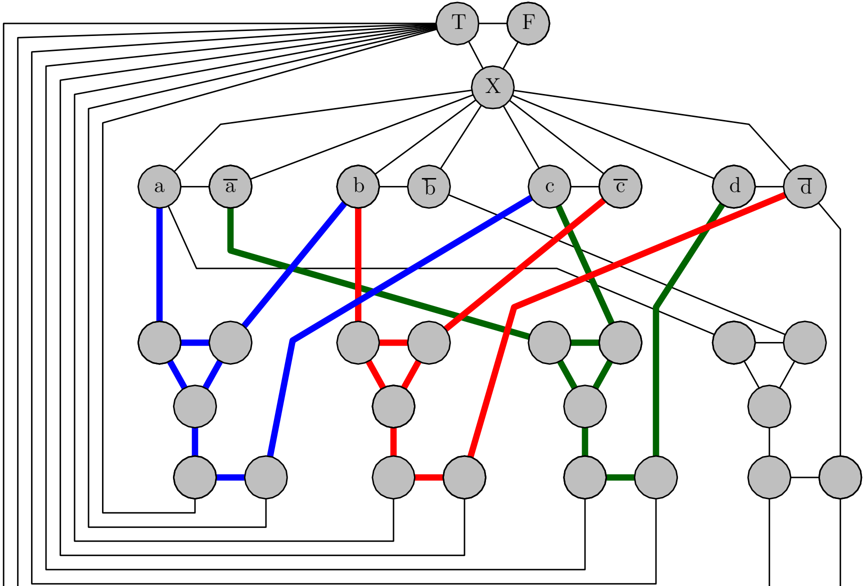
$(a \lor b \lor c) \land (b \lor \overline{c} \lor \overline{d}) \land (\overline{a} \lor c \lor d) \land (a \lor \overline{b} \lor \overline{d})$



Graph generated in reduction...

... from 3SAT to 3COLOR

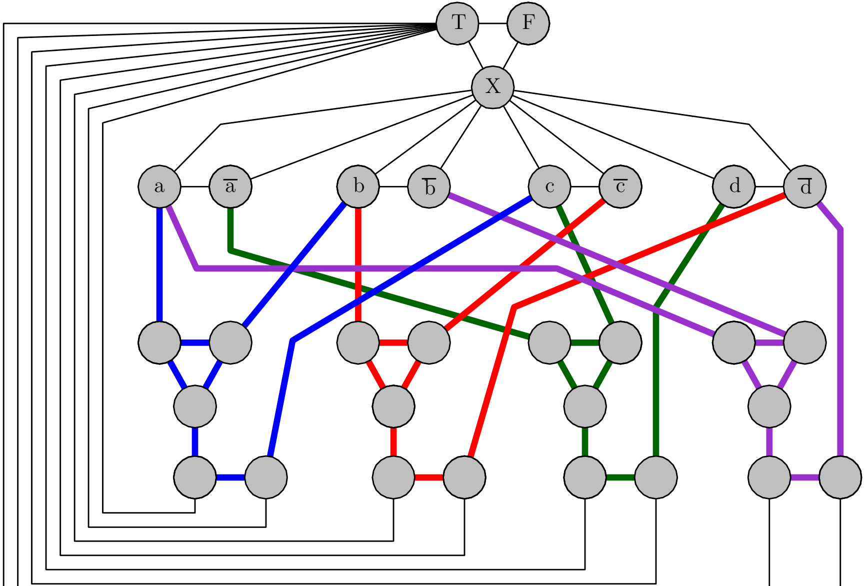
$$(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$$



Graph generated in reduction...

... from 3SAT to 3COLOR

$(a \vee b \vee c) \wedge (b \vee \overline{c} \vee \overline{d}) \wedge (\overline{a} \vee c \vee d) \wedge (a \vee \overline{b} \vee \overline{d})$



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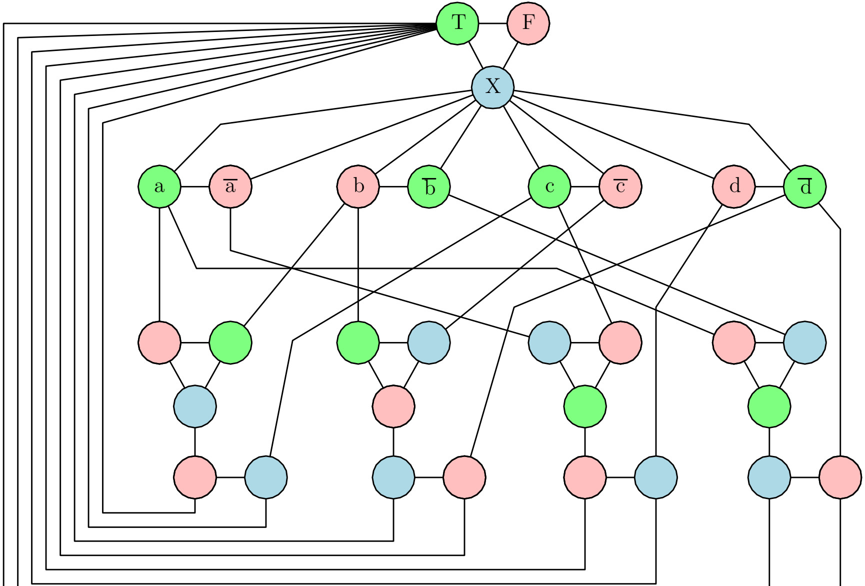
$(a \vee b \vee c) \wedge (b \vee \overline{c} \vee \overline{d}) \wedge (\overline{a} \vee c \vee d) \wedge (a \vee \overline{b} \vee \overline{d})$



THE END

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(for now)