24.3.3
Showing NP-Completeness of **3 COLORING**

24.3.3.1
The variable assignment gadget

3-Coloring is **NP-Complete**

- **3-Coloring** is in **NP**.
  - Certificate: for each node a color from $\{1, 2, 3\}$.
  - Certifier: Check if for each edge (u, v), the color of u is different from that of v.
- Hardness: We will show **3-SAT** $\leq_P$ **3-Coloring**.

Reduction idea

1. φ: Given **3SAT** formula (i.e., **3**CNF formula).
2. φ: variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$.
3. Create graph G_φ s.t. G_φ 3-colorable $\iff$ φ satisfiable.
   - encode assignment $x_1, \ldots, x_n$ in colors assigned nodes of G_φ.
   - create triangle with node True, False, Base
   - for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
   - If graph is 3-colored, either v_i or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to v_i
   - Need to add constraints to ensure clauses are satisfied (next phase)

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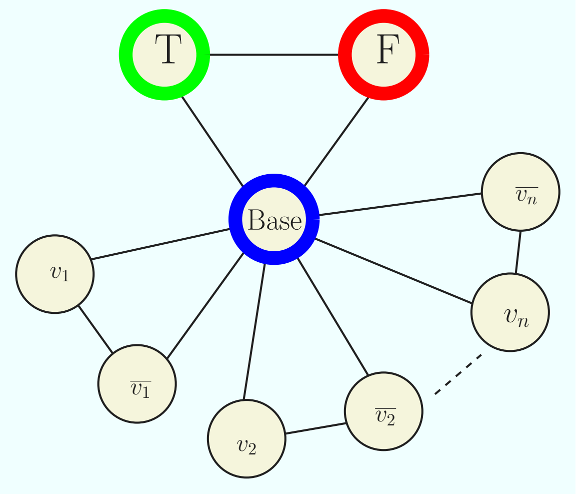
Assignment encoding using **3**-coloring



THE END

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(for now)