23.3
NP-Completeness of Hamiltonian Cycle

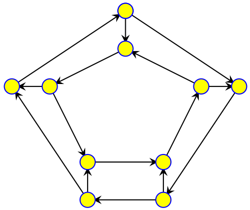
23.3.1
Reduction from 3SAT to Hamiltonian Cycle: Basic idea

Directed Hamiltonian Cycle

Input Given a directed graph $G = (V, E)$ with n vertices

Goal Does G have a Hamiltonian cycle?

- A Hamiltonian cycle is a cycle in the graph that visits every vertex in G exactly once



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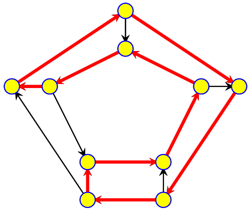
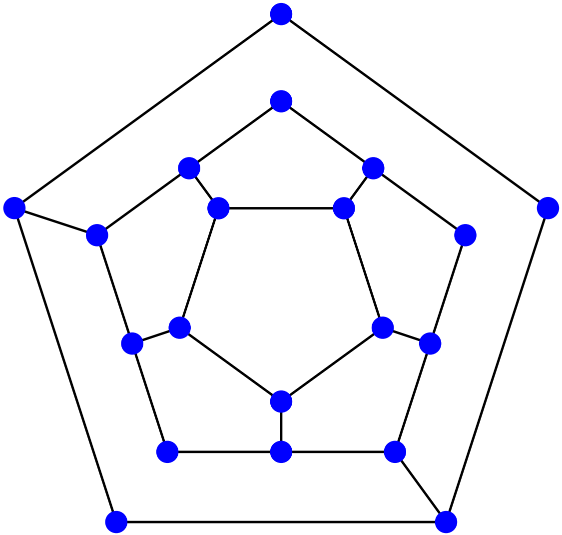


Is the following graph Hamiltonian?



(A) Yes.

(B) No.

Directed Hamiltonian Cycle is **NP-Complete**

- Directed Hamiltonian Cycle is in ***NP***: exercise
- Hardness: We will show **3SAT** $\leq_P$ **Directed Hamiltonian Cycle** .

Reduction construction

From 3SAT to Hamiltonian cycle in directed graph

1. To show reduction, we next describe an algorithm:
   - Input: **3SAT** formula φ
   - Output: A graph G_φ.
   - Running time is polynomial.
   - Requirement: φ is satisfiable $\iff$ G_φ is Hamiltonian.
2. Given **3SAT** formula φ create a graph G_φ such that
   - G_φ has a Hamiltonian cycle if and only if φ is satisfiable
   - G_φ should be constructible from φ by a polynomial time algorithm $\mathcal{A}$
3. Notation: φ has n variables $x_1, x_2, \ldots, x_n$ and m clauses $C_1, C_2, \ldots, C_m$.

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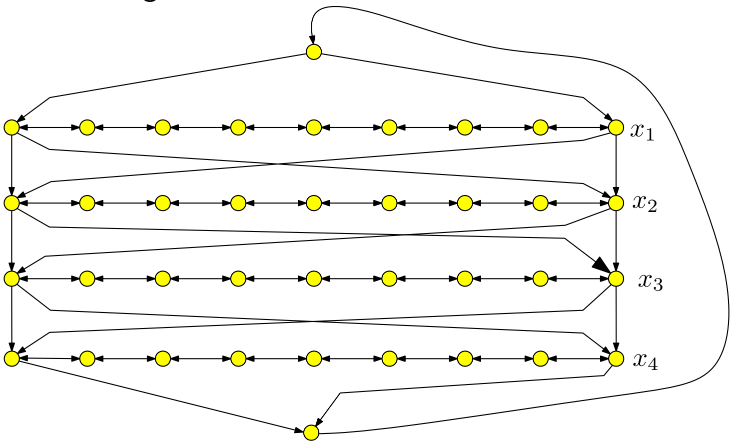
Encoding assignments

Converting φ to a graph

Given a formula with n variables, we need a graph with 2^n different Hamiltonian paths, that can encode their assignments.



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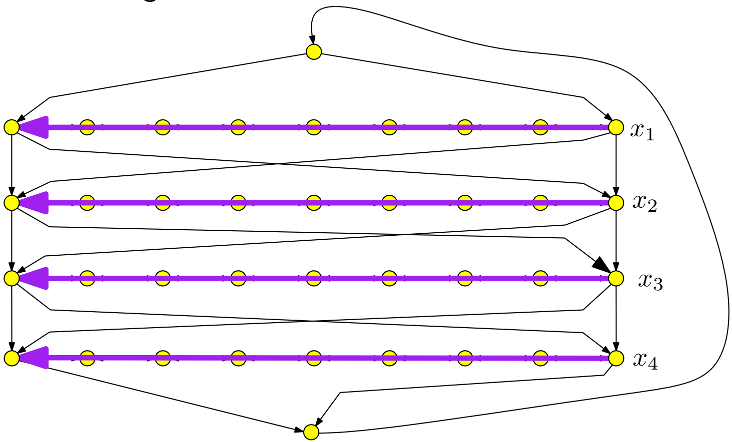

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$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$$

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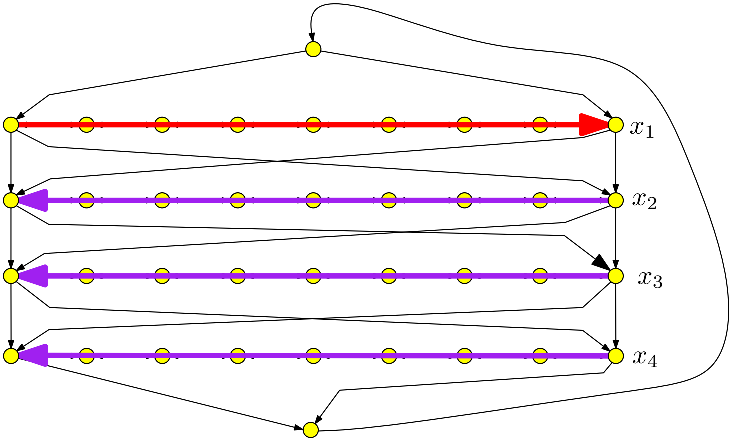

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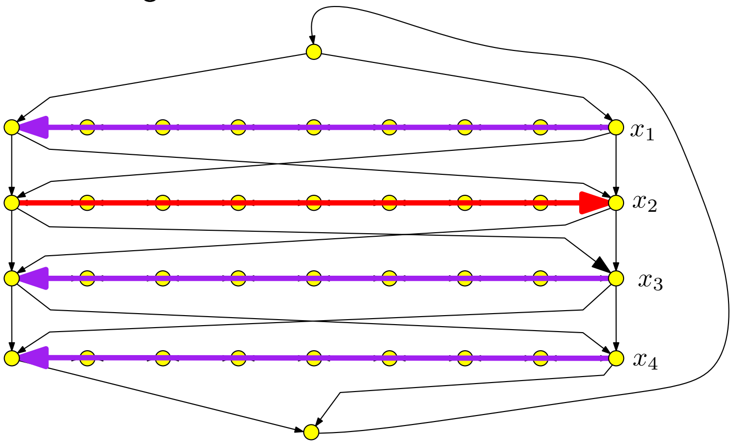

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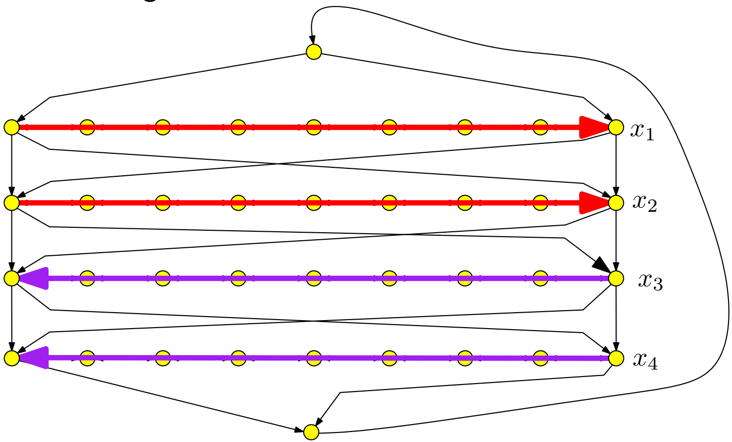

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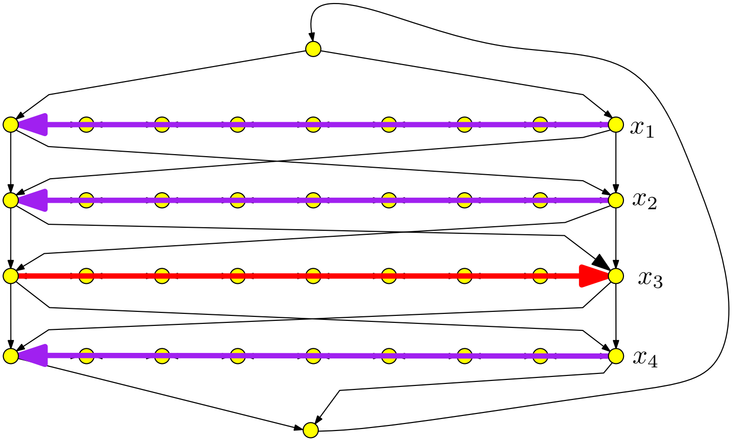

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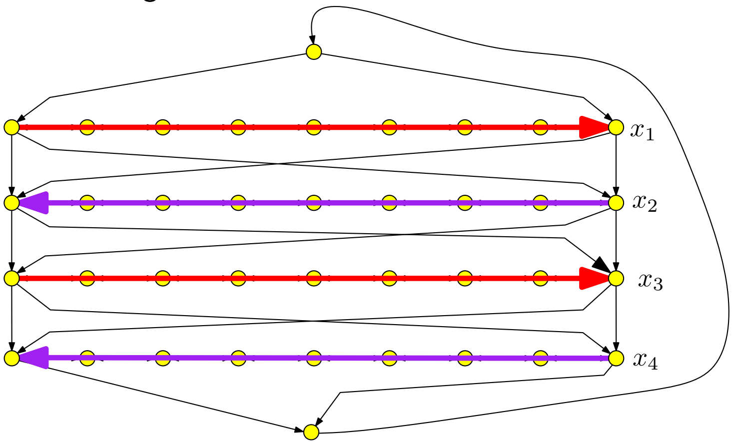

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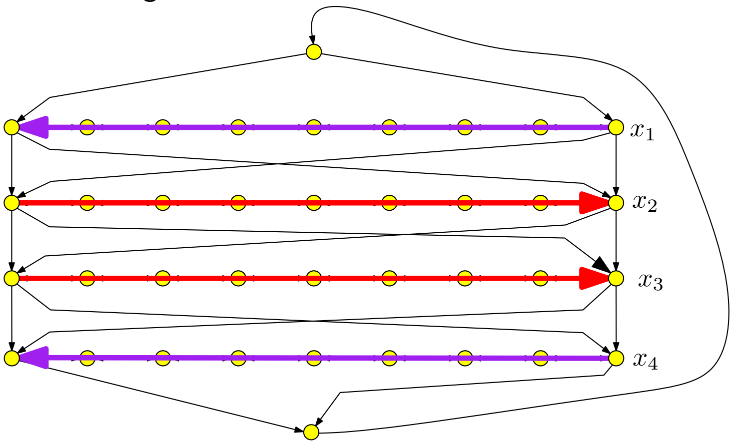

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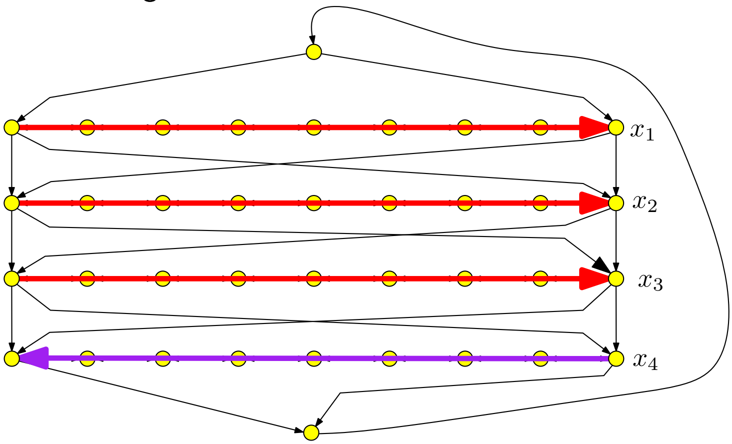

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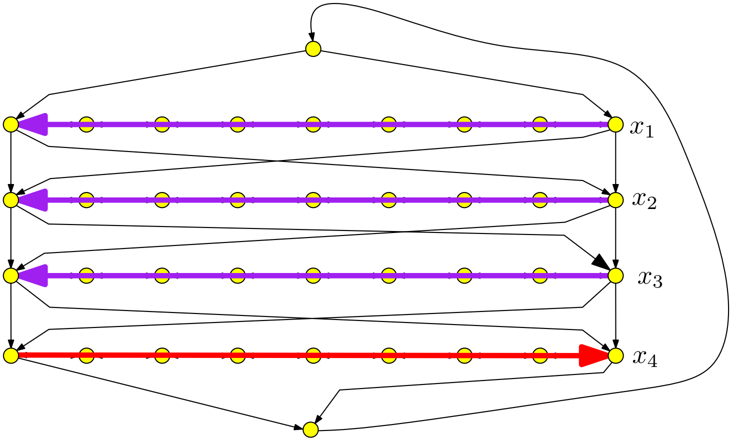

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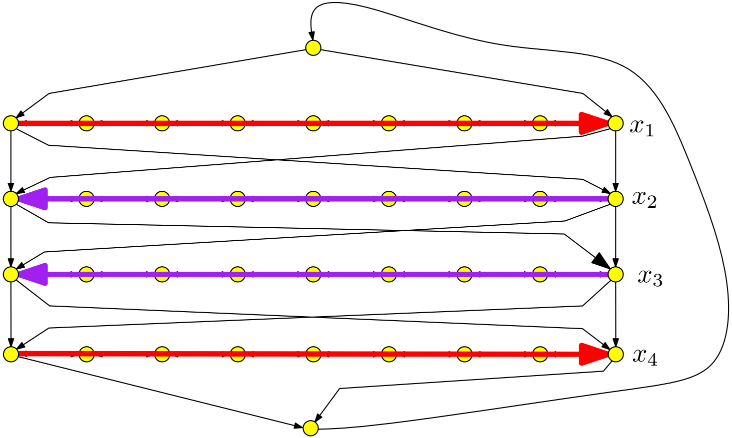

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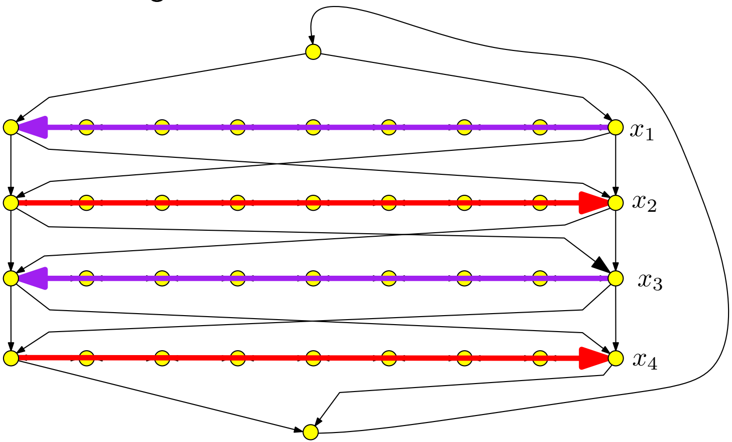

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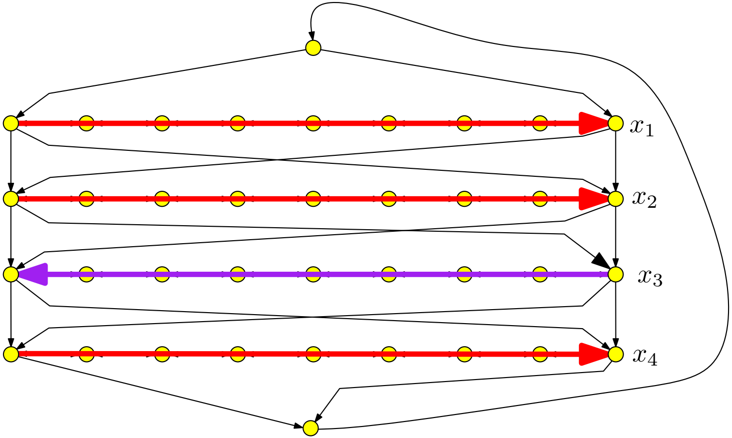

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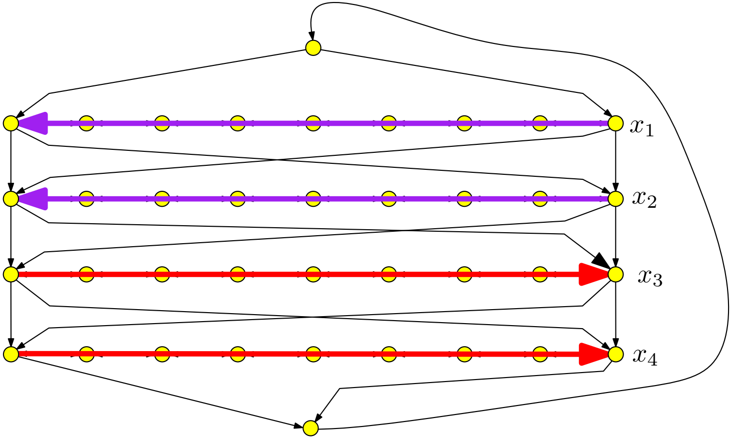

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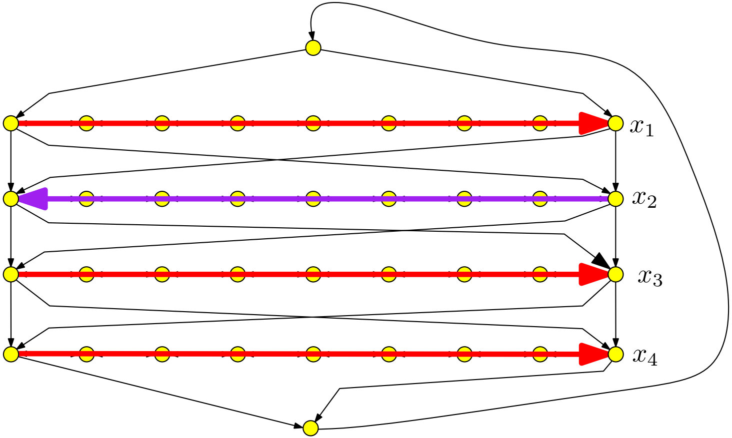

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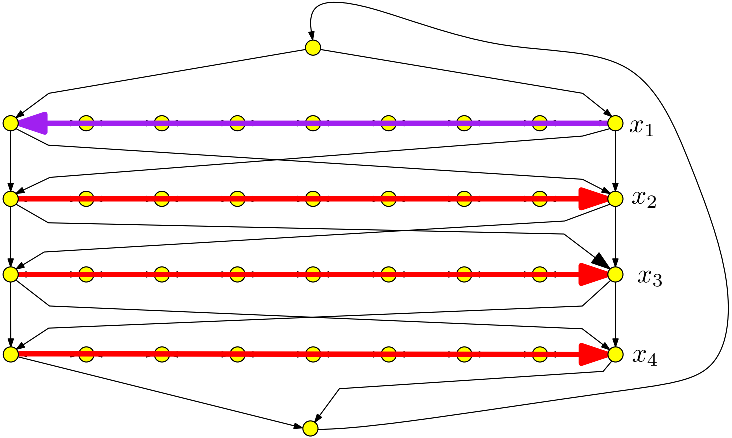

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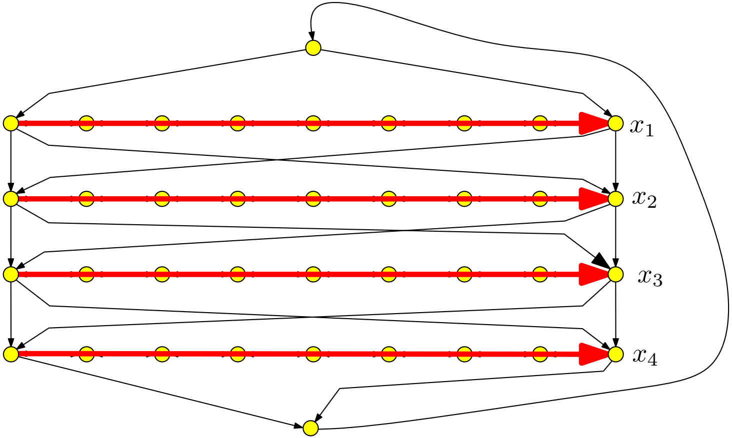

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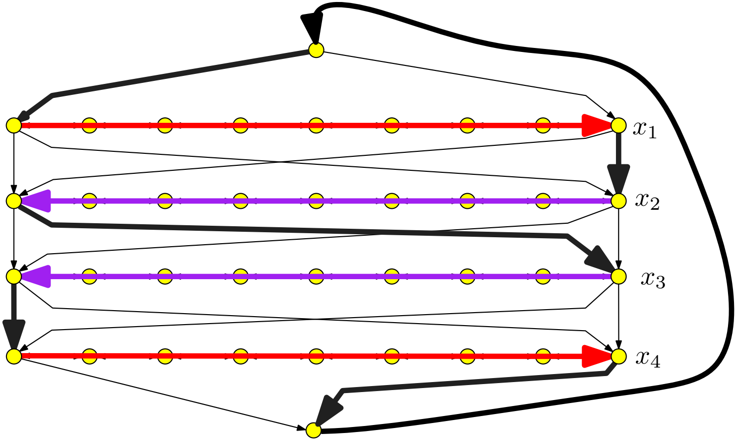

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THE END

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(for now)