21.5
Independent Set and Vertex Cover

Vertex Cover

Given a graph $G = (V, E)$, a set of vertices S is:

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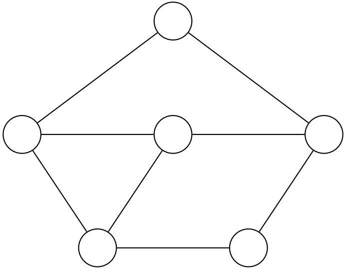

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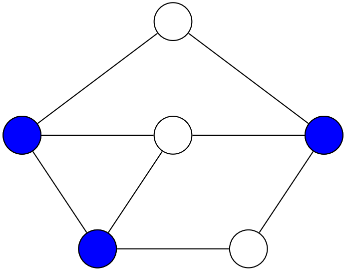

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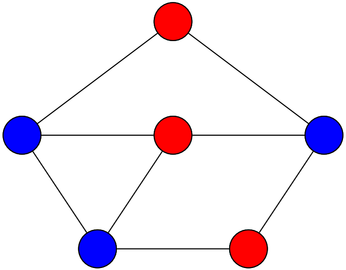

The **Vertex Cover** Problem

Problem 21.1 (Vertex Cover).

Input: *A graph G and integer $\boldsymbol{k}$.*
Goal: *Is there a vertex cover of size $\leq \boldsymbol{k}$ in G?*

Can we relate **Independent Set** and **Vertex Cover**?

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Relationship between...

Vertex Cover and Independent Set

Proposition 21.2.

Let $G = (V, E)$ be a graph. S is an Independent Set $\iff$ $V \setminus S$ is a vertex cover.

Proof.

($\Rightarrow$) Let S be an independent set

1. Consider any edge $uv \in E$.
2. Since S is an independent set, either $u \notin S$ or $v \notin S$.
3. Thus, either $u \in V \setminus S$ or $v \in V \setminus S$.
4. $V \setminus S$ is a vertex cover.

($\Leftarrow$) Let $V \setminus S$ be some vertex cover:

1. Consider $u, v \in S$
2. uv is not an edge of G, as otherwise $V \setminus S$ does not cover uv.
3. $\implies$ S is thus an independent set.

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Independent Set $\leq_P$ Vertex Cover

1. G: graph with n vertices, and an integer k be an instance of the **Independent Set** problem.
2. G has an independent set of size $\geq k \iff G$ has a vertex cover of size $\leq n - k$
3. (G, k) is an instance of **Independent Set**, and $(G, n-k)$ is an instance of **Vertex Cover** with the same answer.
4. Therefore, **Independent Set** $\leq_P$ **Vertex Cover**. Also **Vertex Cover** $\leq_P$ **Independent Set**.

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Proving Correctness of Reductions

To prove that $X \leq_P Y$ you need to give an algorithm $\mathcal{A}$ that:

1. Transforms an instance I_X of X into an instance I_Y of Y.
2. Satisfies the property that answer to I_X is YES $\iff$ I_Y is YES.
    1. typical easy direction to prove: answer to I_Y is YES if answer to I_X is YES
    2. typical difficult direction to prove: answer to I_X is YES if answer to I_Y is YES (equivalently answer to I_X is NO if answer to I_Y is NO).
3. Runs in **polynomial** time.

THE END

...

(for now)