21.3
Examples of Reductions

21.3.1
Independent Set and Clique

Independent Sets and Cliques

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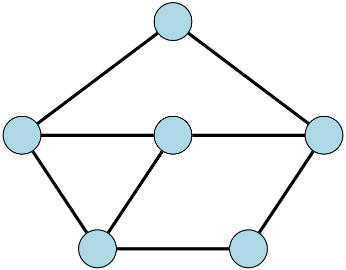

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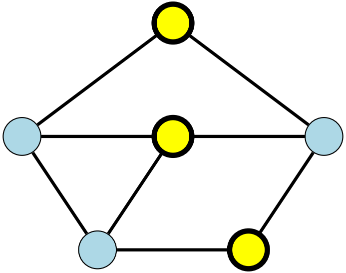

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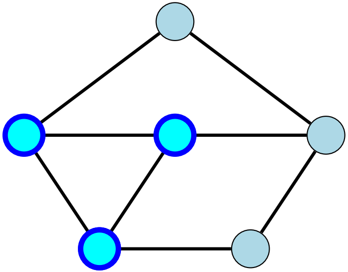

The **Independent Set** and **Clique** Problems

Problem: Independent Set

Instance: A graph G and an integer k. **Question:** Does G has an independent set of size $\geq k$?

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> **Instance:** A graph G and an integer k.
> **Question:** Does G has a clique of size $\geq k$?

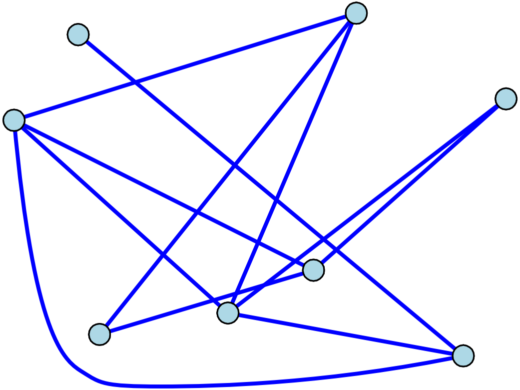
Recall

For decision problems X, Y, a reduction from X to Y is:

1. An algorithm ...
2. that takes I_X, an instance of X as input ...
3. and returns I_Y, an instance of Y as output ...
4. such that the solution (YES/NO) to I_Y is the same as the solution to I_X.

Reducing **Independent Set** to **Clique**

An instance of **Independent Set** is a graph G and an integer k.



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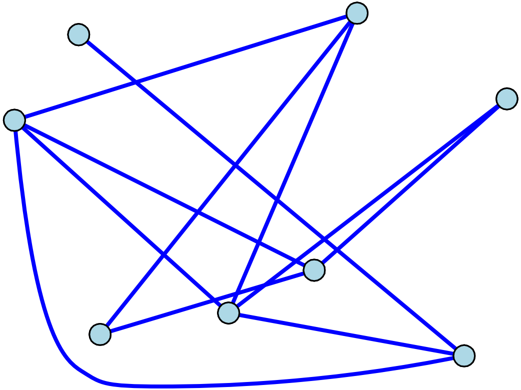

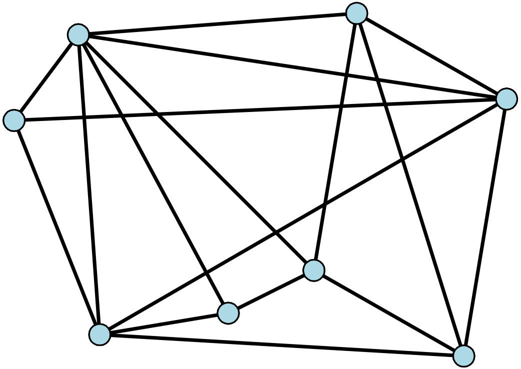

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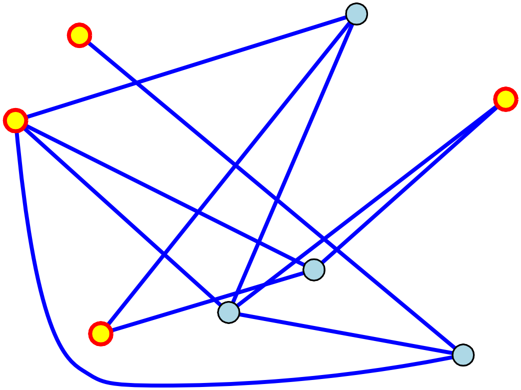
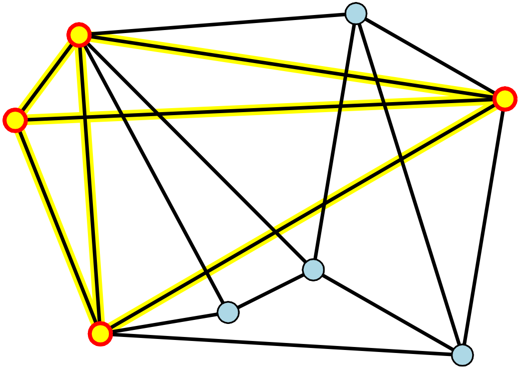
An instance of **Independent Set** is a graph G and an integer k.

Reduction given $\langle G, k \rangle$ outputs $\langle \overline{G}, k \rangle$ where $\overline{G}$ is the complement of G. $\overline{G}$ has an edge $uv \iff uv$ is not an edge of G.





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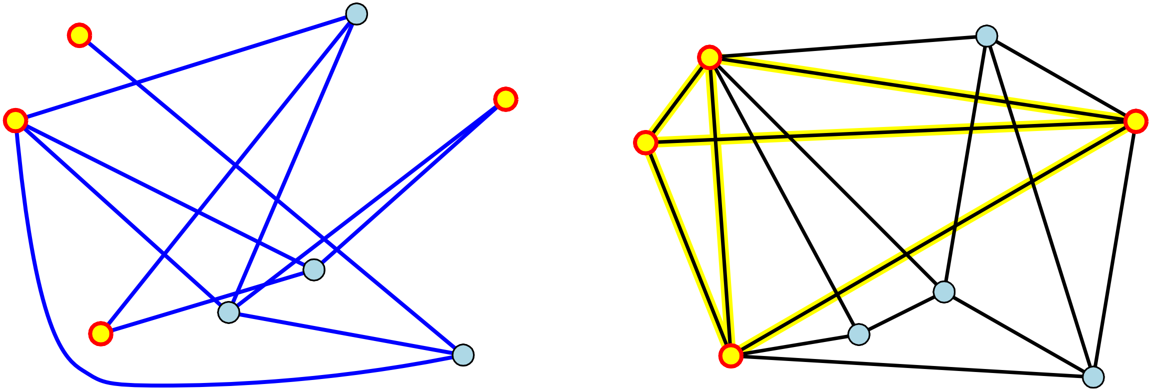

A independent set of size k in $G \iff$ A clique of size k in $\overline{G}$

Correctness of reduction

Lemma 21.1.

G has an independent set of size k $\iff$ $\overline{G}$ has a clique of size k.

Proof.

Need to prove two facts:

G has independent set of size at least k implies that $\overline{G}$ has a clique of size at least k.

$\overline{G}$ has a clique of size at least k implies that G has an independent set of size at least k.

Since $S \subseteq V$ is an independent set in G $\iff$ S is a clique in $\overline{G}$. $\square$

Independent Set and Clique

1. **Independent Set** $\leq$ **Clique**.
   What does this mean?
2. If have an algorithm for **Clique**, then we have an algorithm for **Independent Set**.
3. **Clique** is at least as hard as **Independent Set**.
4. Also... **Clique** $\leq$ **Independent Set**. Why? Thus **Clique** and **Independent Set** are polnomial-time equivalent.

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Review: Independent Set and Clique

Assume you can solve the **Clique** problem in $T(n)$ time. Then you can solve the **Independent Set** problem in

(A) $O(T(n))$ time.

(B) $O(n \log n + T(n))$ time.

(C) $O(n^2 T(n^2))$ time.

(D) $O(n^4 T(n^4))$ time.

(E) $O(n^2 + T(n^2))$ time.

(F) Does not matter - all these are polynomial if $T(n)$ is polynomial, which is good enough for our purposes.

THE END

...

(for now)