20.2
Safe and unsafe edges

Assumption

And for now ...

Assumption 20.1.

Edge costs are distinct, that is no two edge costs are equal.

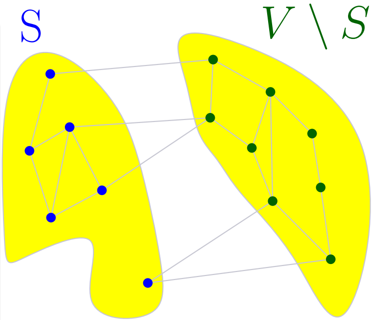
Cuts

Definition 20.2.

Given a graph $G = (V, E)$, a **cut** is a partition of the vertices of the graph into two sets $(S, V \setminus S)$.

Edges having an endpoint on both sides are the **edges of the cut**.

A cut edge is **crossing** the cut.



$$(S, V \setminus S) = \{uv \in E \mid u \in S, v \in V \setminus S\}.$$

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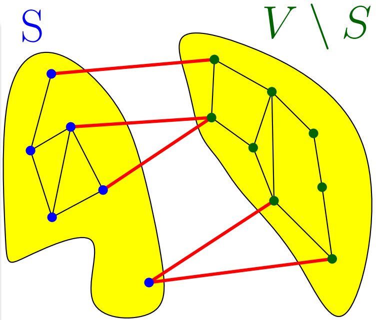

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Safe and Unsafe Edges

Definition 20.3.

An edge $e = (u, v)$ is a safe edge if there is some partition of V into S and $V \setminus S$ and e is the unique minimum cost edge crossing S (one end in S and the other in $V \setminus S$).

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An edge $e = (u, v)$ is an unsafe edge if there is some cycle C such that e is the unique maximum cost edge in C.

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Every edge is either safe or unsafe

Proposition 20.5.

If edge costs are distinct then every edge is either safe or unsafe.

Proof.

Consider any edge $e = uv$.

Let $G_{<w(e)} = (V, \{xy \in E \mid w(xy) < w(e)\})$.

Observe that $e \notin E(G_{<w(e)})$.

1. If x, y in some connected component of $G_{<w(e)}$, then $G_{<w(e)} + e$ contains a cycle where e is most expensive.
   $\implies$ e is unsafe.
2. If x and y are in diff connected component of $G_{<w(e)}$,
   Let S the vertices of connected component of $G_{<w(e)}$ containing x.
   The edge e is cheapest edge in cut $(S, V \setminus S)$.
   $\implies$ e is safe.

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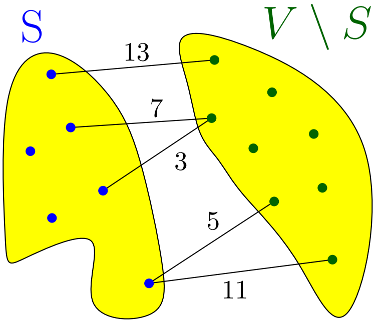
$\square$

Safe edge

Example...

Every cut identifies one safe edge...



...the cheapest edge in the cut.

Note: An edge e may be a safe edge for many cuts!

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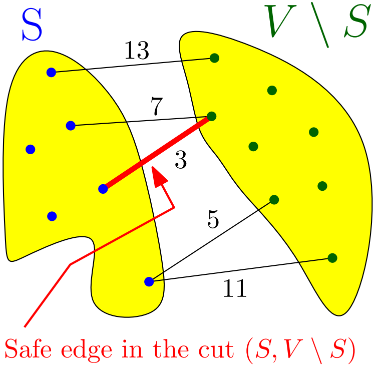

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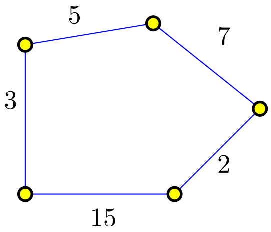

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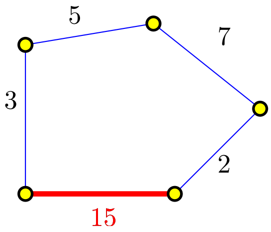

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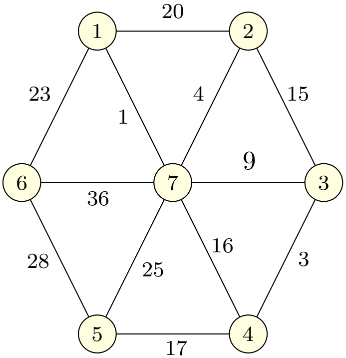

Figure: Graph with unique edge costs. Safe edges are red, rest are unsafe.

And all safe edges are in the MST in this case...

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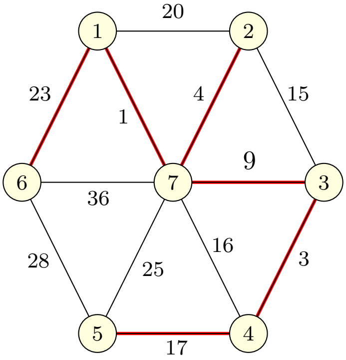

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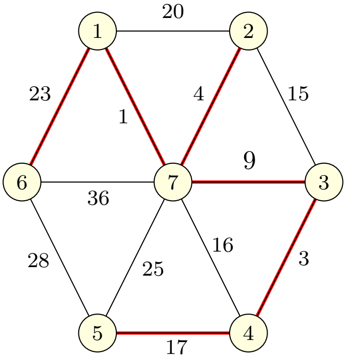

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Some key observations

Proofs later

Lemma 20.6.

If e is a safe edge then **every** *minimum spanning tree contains e.*

Lemma 20.7.

If e is an unsafe edge then no MST of G contains e.

THE END

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(for now)