17.3.4
On the hereditary nature of shortest paths

You can not shortcut a shortest path

Lemma

G: directed graph with non-negative edge lengths.
$\mathrm{dist}(s, v)$: shortest path length from s to v.
If $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k$ shortest path from s to v_k then for any $0 \le i < j \le k$:
$v_i \to v_{i+1} \to \ldots \to v_j$ is shortest path from v_i to v_j

Proof.

Suppose not. Then for some $0 \le i < j \le k$ there is a path P' from v_i to v_j of length strictly less than that of $s = v_i \to v_{i+1} \to \ldots \to v_j$. Then the path

$$s = v_0 \to v_1 \to \cdots \to v_i \bullet P' \bullet v_j \to v_{j+1} \to \cdots \to v_k$$

is a strictly shorter path from s to v_k than $s = v_0 \to v_1 \ldots \to v_k$. □

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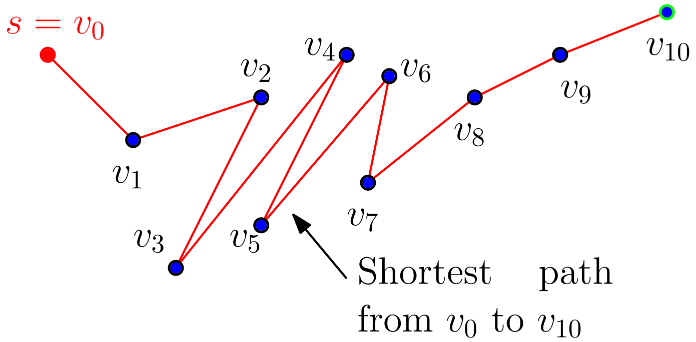
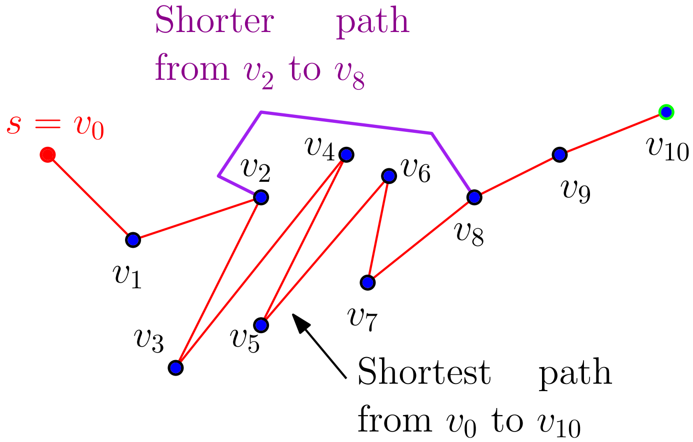
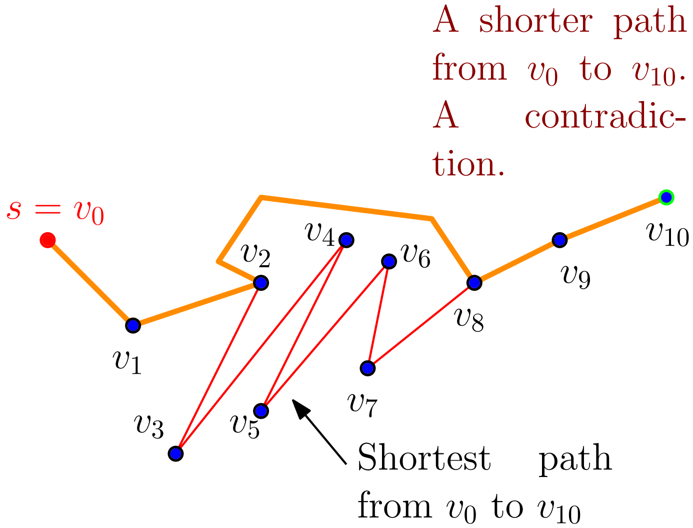
A proof by picture



A proof by picture



A proof by picture



What we really need...

Corollary

G: directed graph with non-negative edge lengths.
$\text{dist}(s, v)$*: shortest path length from s to v.*
If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ shortest path from s to v_k then for any $0 \leq i \leq k$:

1. *$s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is shortest path from s to v_i*
2. $\text{dist}(s, v_i) \leq \text{dist}(s, v_k)$. *Relies on non-neg edge lengths.*

THE END

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(for now)