### Algorithms & Models of Computation

CS/ECE 374, Fall 2020

# 17.3.4

On the hereditary nature of shortest paths

### You can not shortcut a shortest path

#### Lemma

**G**: directed graph with non-negative edge lengths.

dist(s, v): shortest path length from s to v.

If  $s = \mathbf{v}_0 \to \mathbf{v}_1 \to \mathbf{v}_2 \to \ldots \to \mathbf{v}_k$  shortest path from s to  $\mathbf{v}_k$  then for any

 $0 \le i < j \le k$ :

 $\mathbf{v}_i 
ightarrow \mathbf{v}_{i+1} 
ightarrow \ldots 
ightarrow \mathbf{v}_i$  is shortest path from  $\mathbf{v}_i$  to  $\mathbf{v}_j$ 

#### Proof

Suppose not. Then for some  $0 \le i < j \le k$  there is a path P' from  $v_i$  to  $v_j$  of length strictly less than that of  $s = v_i \to v_{i+1} \to \ldots \to v_j$ . Then the path

$$s = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_i \bullet P' \bullet v_j \rightarrow v_{j+1} \rightarrow \cdots \rightarrow v_k$$

is a strictly shorter path from s to  $v_k$  than  $s = v_0 \rightarrow v_1 \ldots \rightarrow v_k$ .

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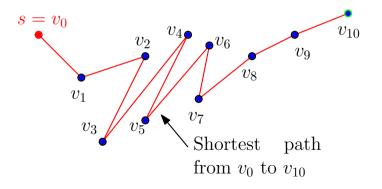
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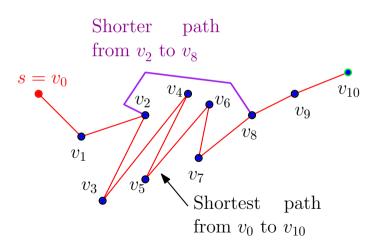
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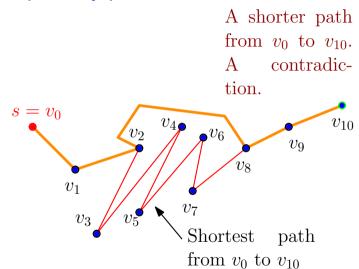
## A proof by picture



## A proof by picture



### A proof by picture



### What we really need...

### Corollary

**G**: directed graph with non-negative edge lengths.

dist(s, v): shortest path length from s to v.

If  $s= extbf{v}_0 o extbf{v}_1 o extbf{v}_2 o \ldots o extbf{v}_k$  shortest path from s to  $extbf{v}_k$  then for any

- $0 \le i \le k$ :
  - $lackbox{0} \quad s = m{v}_0 
    ightarrow m{v}_1 
    ightarrow m{v}_2 
    ightarrow \ldots 
    ightarrow m{v}_i$  is shortest path from s to  $m{v}_i$
  - $\bigcirc$  dist $(s, v_i) \le \text{dist}(s, v_k)$ . Relies on non-neg edge lengths.

# THE END

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(for now)