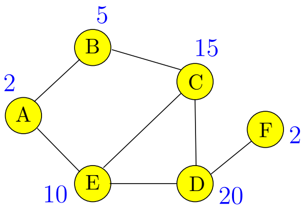
14.3
Maximum Weighted Independent Set in Trees

Maximum Weight Independent Set Problem

Input Graph $G = (V, E)$ and weights $w(v) \geq 0$ for each $v \in V$

Goal Find maximum weight independent set in G



Maximum weight independent set in above graph: $\{B, D\}$

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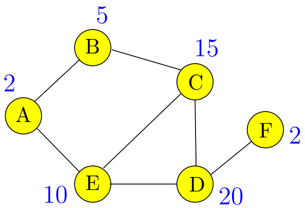
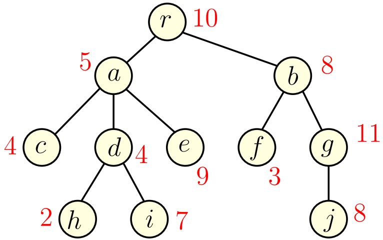


Maximum weight independent set in above graph: $\{B, D\}$

Maximum Weight Independent Set in a Tree

Input Tree $T = (V, E)$ and weights $w(v) \geq 0$ for each $v \in V$

Goal Find maximum weight independent set in T



Maximum weight independent set in above tree: ??

Towards a Recursive Solution

For an arbitrary graph G:

1. Number vertices as $v_1, v_2, \ldots, v_n$
2. Find recursively optimum solutions without v_n (recurse on $G - v_n$) and with v_n (recurse on $G - v_n - N(v_n)$ & include v_n).
3. Saw that if graph G is arbitrary there was no good ordering that resulted in a small number of subproblems.

What about a tree? Natural candidate for v_n is root r of T?

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Towards a Recursive Solution

Natural candidate for v_n is root r of T? Let $\mathcal{O}$ be an optimum solution to the whole problem.

Case $r \notin \mathcal{O}$: Then $\mathcal{O}$ contains an optimum solution for each subtree of T hanging at a child of r.

Case $r \in \mathcal{O}$: None of the children of r can be in $\mathcal{O}$. $\mathcal{O} - \{r\}$ contains an optimum solution for each subtree of T hanging at a grandchild of r.

Subproblems? Subtrees of T rooted at nodes in T.

How many of them? $O(n)$

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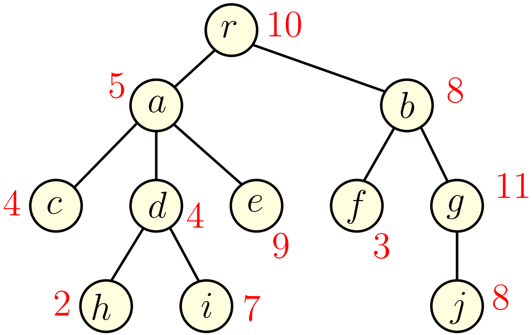
Example



A Recursive Solution

$\boldsymbol{T(u)}$: subtree of $\boldsymbol{T}$ hanging at node $\boldsymbol{u}$
$\boldsymbol{OPT(u)}$: max weighted independent set value in $\boldsymbol{T(u)}$

$$\boldsymbol{OPT(u)} = \max \begin{cases} \sum_{v \text{ child of } u} OPT(v), \\ w(u) + \sum_{v \text{ grandchild of } u} OPT(v) \end{cases}$$

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Iterative Algorithm

1. Compute $OPT(u)$ bottom up. To evaluate $OPT(u)$ need to have computed values of all children and grandchildren of u
2. What is an ordering of nodes of a tree T to achieve above? Post-order traversal of a tree.

Iterative Algorithm

1. Compute $\boldsymbol{OPT}(\boldsymbol{u})$ bottom up. To evaluate $\boldsymbol{OPT}(\boldsymbol{u})$ need to have computed values of all children and grandchildren of $\boldsymbol{u}$
2. What is an ordering of nodes of a tree $\boldsymbol{T}$ to achieve above? Post-order traversal of a tree.

Iterative Algorithm

MIS-Tree(T):
    Let $v_1, v_2, \ldots, v_n$ be a post-order traversal of nodes of T
    for $i = 1$ to n **do**
        $$M[v_i] = \max\begin{pmatrix} \sum_{v_j \text{ child of } v_i} M[v_j], \\ w(v_i) + \sum_{v_j \text{ grandchild of } v_i} M[v_j] \end{pmatrix}$$
    return $M[v_n]$ (* Note: v_n is the root of T *)

Space: $O(n)$ to store the value at each node of T

Running time:

1. Naive bound: $O(n^2)$ since each $M[v_i]$ evaluation may take $O(n)$ time and there are n evaluations.
2. Better bound: $O(n)$. A value $M[v_j]$ is accessed only by its parent and grand parent.

Iterative Algorithm

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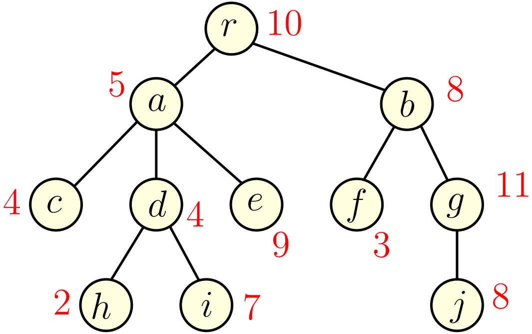

THE END

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(for now)