# Algorithms & Models of Computation CS/ECE 374, Fall 2020

**9.4** Unrecognizable

#### **Definition**

Language  $\boldsymbol{L}$  is TM decidable if there exists  $\boldsymbol{M}$  that always stops, such that  $\boldsymbol{L}(\boldsymbol{M}) = \boldsymbol{L}$ .

#### Definition

Language L is TM recognizable if there exists M that stops on some inputs, such that L(M) = L.

### Theorem (Halting)

 $\mathbf{A}_{\mathrm{TM}} = \Big\{ \langle \mathbf{M}, \mathbf{w} 
angle \ | \ \mathbf{M} \ \text{is a TM} \ \text{and} \ \mathbf{M} \ \text{accepts} \ \mathbf{w} \Big\}$  . is  $\mathrm{TM} \ \text{recognizable}$ , but not decidable.

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Language  ${f L}$  is  ${f TM}$  <u>decidable</u> if there exists  ${f M}$  that always stops, such that  ${f L}({f M})={f L}$ .

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### Theorem (Halting)

#### Lemma

If **L** and  $\overline{\mathbf{L}} = \Sigma^* \setminus \mathbf{L}$  are both TM recognizable, then **L** and  $\overline{\mathbf{L}}$  are decidable.

#### Proof.

M: TM recognizing L

 $M_c$ : TM recognizing  $\overline{L}$ .

Given input x, using UTM simulating running M and  $M_c$  on x in parallel. One of them must stop and accept. Return result.

 $\implies$  L is decidable.

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### Complement language for A<sub>TM</sub>

$$\overline{\mathbf{A}_{\mathrm{TM}}} = \Sigma^* \setminus \left\{ \langle extbf{ extit{M}}, extbf{ extit{w}} 
angle \ ig| extbf{ extit{M}} ext{ is a } \mathbf{TM} ext{ and } extbf{ extit{M}} ext{ accepts } extbf{ extit{w}} 
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But don't really care about invalid inputs. So, really:

$$\overline{\mathbf{A}_{\mathrm{TM}}} = \Big\{ \langle oldsymbol{M}, oldsymbol{w} 
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# Complement language for A<sub>TM</sub> is not TM-recognizable

#### Theorem

The language

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is not TM recognizable.

#### Proof.

A<sub>TM</sub> is TM-recognizable.

If  $\overline{\mathbf{A}_{\mathrm{TM}}}$  is TM-recognizable

⇒ (by Lemma)

**A**<sub>TM</sub> is decidable. A contradiction.

# Complement language for $A_{TM}$ is not TM-recognizable

#### Theorem

The language

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is not TM recognizable.

#### Proof.

 $A_{\rm TM}$  is  ${\rm TM}$ -recognizable.

If  $\overline{\mathbf{A}_{\mathrm{TM}}}$  is  $\mathrm{TM}$ -recognizable

⇒ (by Lemma

A<sub>TM</sub> is decidable. A contradiction.

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The language

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 $A_{\rm TM}$  is  ${\rm TM}$ -recognizable.

If  $\overline{\mathbf{A}_{\mathrm{TM}}}$  is  $\mathrm{TM}$ -recognizable

 $\implies$  (by Lemma)

**A**<sub>TM</sub> is decidable. A contradiction.

# THE END

...

(for now)