Prove that each of the following languages is not regular.

1. \( \{0^{2^n} \mid n \geq 0\} \)

**Solution (\(F = L\)):** Let \( F = L = \{0^{2^n} \mid n \geq 0\} \).

Let \( x \) and \( y \) be arbitrary distinct elements of \( F \).

Then \( x = 0^{2^i} \) and \( y = 0^{2^j} \) for some non-negative integers \( i \neq j \).

Let \( z = 0^{2^k} \).

- \( xz = 0^{2^i}0^{2^k} = 0^{2^{k+1}} \in L \).
- \( yz = 0^{2^j}0^{2^k} = 0^{2^{k+2}} \not\in L \), because \( i \neq j \), and thus \( 2^k + 2^i \) is not a power of 2.

Because \( xz \in L \) and \( yz \not\in L \), the suffix \( z \) distinguishes \( x \) and \( y \).

But \( x \) and \( y \) were arbitrary, so every pair of elements in \( F \) has a distinguishing suffix.

In other words, \( F \) is a fooling set for \( L \).

We conclude that \( L \) cannot be regular, because the fooling set \( F \) is infinite.

**Solution (\(F = 0^*\)):** Let \( F = 0^* = \{0^n \mid n \geq 0\} \).

Let \( x \) and \( y \) be arbitrary distinct elements of \( F \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Without loss of generality, assume \( i < j \).

Let \( r \) be any integer such that \( 2^r > j \), and let \( z = 0^{2^r-i} \).

Then \( xz = 0^i0^{2^r-i} = 0^{2^r} \in L \).

But \( yz = 0^j0^{2^r-i} = 0^{2^r+j-i} \not\in L \), because \( 2^r + j - i \) is not a power of 2:

\[
2^r < 2^r + j - i < 2^r + j < 2^r + 2^r = 2^{r+1}
\]

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

**Solution (\(F = 0^*, \) but distinguish the other way):** Let \( F = 0^* = \{0^n \mid n \geq 0\} \).

Let \( x \) and \( y \) be arbitrary distinct elements of \( F \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Without loss of generality, assume \( i < j \).

Let \( r \) be any integer such that \( 2^{r-1} > j \), and let \( z = 0^{2^r-j} \).

Then \( xz = 0^i0^{2^r-j} = 0^{2^r-j+i} \not\in L \), because \( 2^r - j + i \) is not a power of 2:

\[
2^{r-1} = 2^r - 2^r - 1 < 2^r - j < 2^r - j + i < 2^r 
\]

But \( yz = 0^j0^{2^r-j} = 0^{2^r} \in L \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.
2. \{0^{2n+1}1^n \mid n \geq 0\}

**Solution \((F = (\emptyset \emptyset)^*)\):** Let \(F\) be the language \((\emptyset \emptyset)^*\).

Let \(x\) and \(y\) be arbitrary distinct strings in \(F\).

Then \(x = \emptyset^{2i}\) and \(y = \emptyset^{2j}\) for some non-negative integers \(i \neq j\).

Let \(z = 1^i\).

Then \(xz = \emptyset^{2i+1} \in L\).

And \(yz = \emptyset^{2j+1} \notin L\), because \(i \neq j\).

Thus, \(F\) is a fooling set for \(L\).

Because \(F\) is infinite, \(L\) cannot be regular. ■

**Solution \((F = \emptyset^*)\):** Let \(F\) be the language \(\emptyset^*\).

Let \(x\) and \(y\) be arbitrary distinct strings in \(F\).

Then \(x = \emptyset^{i}\) and \(y = \emptyset^{j}\) for some non-negative integers \(i \neq j\).

Let \(z = 0^i 1^i\).

Then \(xz = \emptyset^{2i+1} \in L\).

And \(yz = \emptyset^{i+j+1} \notin L\), because \(i + j \neq 2i\).

Thus, \(F\) is a fooling set for \(L\).

Because \(F\) is infinite, \(L\) cannot be regular. ■

**Solution (via homomorphism):** Suppose to the contrary that \(L\) is the language of some DFA \(M = (Q, s, A, \delta)\). Construct a new DFA \(M' = (Q, s, A, \delta')\) with the same states, start state, and accepting states as \(M\), but with a new transition function:

\[
\delta'(q, a) = \begin{cases} 
\delta^*(q, \emptyset \emptyset) & \text{if } a = \emptyset \\
\delta(q, 1) & \text{if } a = 1
\end{cases}
\]

In other words, \(M'\) simulates \(M\), but pretends that every \(\emptyset\) it reads is actually two \(\emptyset\)s.

Let \(\text{doubleoh}\) be the following string function:

\[
\text{doubleoh}(w) := \begin{cases} 
\varepsilon & \text{if } w = \varepsilon \\
\emptyset \emptyset \cdot \text{doubleoh}(x) & \text{if } w = \emptyset x \\
1 \cdot \text{doubleoh}(x) & \text{if } w = 1x
\end{cases}
\]

In particular, for any integer \(n\), we have \(\text{doubleoh}(\emptyset^n 1^n) = \emptyset^{2n+1} 1^n\). Straightforward but tedious induction implies that our new DFA \(M'\) accepts a string \(w\) if and only if the original DFA \(M\) accepts the string \(\text{doubleoh}(w)\). It follows that \(L(M') = \{\emptyset^n 1^n \mid n \geq 0\}\).

But we proved in class that \(L(M')\) is not regular, so we have reached a contradiction; the original DFA \(M\) cannot exist! ■

[Yes, this proof would be worth full credit, both on homework and exams. But the fooling set argument is simpler, so try that first!]
3. \{0^m1^n \mid m \neq 2n\}

**Solution (F = (\emptyset0)^*)**: Let \(F\) be the language \((\emptyset0)^*\).

Let \(x\) and \(y\) be arbitrary distinct strings in \(F\).

Then \(x = 0^{2i}\) and \(y = 0^{2j}\) for some non-negative integers \(i \neq j\).

Let \(z = 1^i\).

Then \(xz = 0^{2i+1} \not\in L\).

And \(yz = 0^{2j+1} \in L\), because \(i \neq j\).

Thus, \(F\) is a fooling set for \(L\).

Because \(F\) is infinite, \(L\) cannot be regular.

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**Solution (F = \emptyset^*)**: Let \(F\) be the language \(\emptyset^*\).

Let \(x\) and \(y\) be arbitrary distinct strings in \(F\).

Then \(x = 0^i\) and \(y = 0^j\) for some non-negative integers \(i \neq j\).

Let \(z = 0^i1^i\).

Then \(xz = 0^{2i+1} \not\in L\).

And \(yz = 0^{i+j+1} \in L\), because \(i + j \neq 2i\).

Thus, \(F\) is a fooling set for \(L\).

Because \(F\) is infinite, \(L\) cannot be regular.

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**Solution (via closure properties)**: If \(L\) were regular, then the language

\[\emptyset^*1^* \setminus L = \{0^m1^n \mid m = 2n\} = \{0^{2n+1}n \mid n \geq 0\}\]

would also be regular, because regular languages are closed under complement. But we just proved that \(\{0^{2n+1}n \mid n \geq 0\}\) is not regular in problem 2.

[Yes, this proof would be worth full credit, either in homework or on an exam.]
4. Strings over \{0, 1\} where the number of 0s is exactly twice the number of 1s.

**Solution (F = 1*)**: Let F be the language 1*. Let x and y be arbitrary distinct strings in F. Then \(x = 1^i\) and \(y = 1^j\) for some non-negative integers \(i \neq j\). Let \(z = 0^{2i}\). Then \(xz = 1^i0^{2i} \in L\). And \(yz = 1^i0^{2j} \notin L\), because \(i \neq j\). Thus, F is a fooling set for L. Because F is infinite, L cannot be regular.

**Solution (F = 0*)**: Let F be the language 0*. Let x and y be arbitrary distinct strings in F. Then \(x = 0^i\) and \(y = 0^j\) for some non-negative integers \(i \neq j\). Let \(z = 0^i1^i\). Then \(xz = 0^{2i}1^i \in L\). And \(yz = 0^{i+j}1^i \notin L\), because \(i + j \neq 2i\). Thus, F is a fooling set for L. Because F is infinite, L cannot be regular.

**Solution (via closure properties)**: If L were regular, then the language 
\[L \cap 0^*1^* = \{0^{2n}1^n \mid n \geq 0\}\] would also be regular, because regular languages are closed under intersection. But we just proved that \(\{0^{2n}1^n \mid n \geq 0\}\) is not regular in problem 2.

[Yes, this proof would be worth full credit, either in homework or on an exam.]
5. Strings of properly nested parentheses ( ), brackets [ ], and braces { }. For example, the string ( [ ] ) { } is in this language, but the string ( [ ] ) is not, because the left and right delimiters don’t match.

Solution: Let $F$ be the language $(^*)$.

Let $x$ and $y$ be arbitrary distinct strings in $F$.

Then $x = (^i)$ and $y = (^j)$ for some non-negative integers $i \neq j$.

Let $z = (^i)$.

Then $xz = (^i)^i \in L$.

And $yz = (^j)^j \notin L$, because $i \neq j$.

Thus, $F$ is a fooling set for $L$.

Because $F$ is infinite, $L$ cannot be regular.

[Notice that this argument doesn’t even try to consider strings with different types of brackets, because it doesn’t have to. Our language $L$ has an infinite fooling set $F$; that’s enough.]

Solution (via closure properties): If $L$ were regular, then the language

$$L' := L \cap [^*]^* = \{[^n]^n \mid n \geq 0\}$$

would also be regular, because regular languages are closed under intersection. But $L'$ is the same as the language $\{0^n 1^n \mid n \geq 0\}$, except for renaming the symbols $0 \mapsto [ \text{ and } 1 \mapsto ]$, and we proved that $\{0^n 1^n \mid n \geq 0\}$ in class.

[Yes, this proof would be worth full credit, either in homework or on an exam.]
Harder problems to think about later:

6. Strings of the form $w_1#w_2# \cdots #w_n$ for some $n \geq 2$, where each substring $w_i$ is a string in \{0, 1\}*, and some pair of substrings $w_i$ and $w_j$ are equal.

Solution (consider the special case $n = 2$):

Let $F$ be the language $0^*$.  
Let $x$ and $y$ be arbitrary distinct strings in $F$.  
Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.  
Let $z = #0^i$.  
Then $xz = 0^i#0^i \in L$.  
And $yz = 0^i#0^j \notin L$, because $i \neq j$.  
Thus, $F$ is a fooling set for $L$.  
Because $F$ is infinite, $L$ cannot be regular.  

[Notice that this argument doesn’t even try to consider strings with more than one #, because it doesn’t have to. Our language $L$ has an infinite fooling set $F$; that’s enough.]
7. \( \{0^{n^2} \mid n \geq 0\} \)

**Solution \( F = L \):** Let \( x \) and \( y \) be arbitrary distinct strings in \( L \).

Without loss of generality, \( x = 0^{i^2} \) and \( y = 0^{j^2} \) for some \( i > j \geq 0 \).

Let \( z = 0^{2i+1} \).

Then \( xz = 0^{i^2+2i+1} = 0^{(i+1)^2} \in L \).

On the other hand, \( yz = 0^{i^2+2j+1} \not\in L \), because \( i^2 < i^2 + 2j + 1 < (i+1)^2 \).

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( L \) is a fooling set for \( L \).

Because \( L \) is infinite, \( L \) cannot be regular. \( \blacksquare \)

**Solution \( F = 0^{*} \):** Let \( x \) and \( y \) be arbitrary distinct strings in \( 0^{*} \).

Without loss of generality, \( x = 0^i \) and \( y = 0^j \) for some \( i > j \geq 0 \).

Let \( z = 0^{i^2+i+1} \).

Then \( xz = 0^{i^2+2i+1} = 0^{(i+1)^2} \in L \).

On the other hand, \( yz = 0^{i^2+i+j+1} \not\in L \), because \( i^2 < i^2 + i + j + 1 < (i+1)^2 \).

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( 0^{*} \) is a fooling set for \( L \).

Because \( 0^{*} \) is infinite, \( L \) cannot be regular. \( \blacksquare \)

**Solution \( F = 0000^{*} \):** Let \( x \) and \( y \) be arbitrary distinct strings in \( 0000^{*} \).

Without loss of generality, \( x = 0^i \) and \( y = 0^j \) for some \( i > j \geq 3 \).

Let \( z = 0^{i^2-i} \).

Then \( xz = 0^{i^2} \in L \).

On the other hand, \( yz = 0^{i^2-i+j} \not\in L \), because

\[
(i-1)^2 = i^2 - 2i + 1 < i^2 - i < i^2 - i + j < i^2.
\]

(The first inequality requires \( i \geq 2 \), and the second requires \( j \geq 1 \).)

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( 0000^{*} \) is a fooling set for \( L \).

Because \( 0000^{*} \) is infinite, \( L \) cannot be regular. \( \blacksquare \)
8. \{w \in (\{0, 1\})^* \mid w \text{ is the binary representation of a perfect square}\}

\textbf{Solution:} We design our fooling set around numbers of the form \((2^k + 1)^2 = 2^{2k + 2k+1} + 1 = 10^{k-2}10^k1 \in L\), for any integer \(k \geq 2\). The argument is somewhat simpler if we further restrict \(k\) to be even.

Let \(F = 1(00)^*1\), and let \(x\) and \(y\) be arbitrary distinct strings in \(F\). Then \(x = 10^{2i-2}1\) and \(y = 10^{2j-1}\), for some positive integers \(i \neq j\).

Without loss of generality, assume \(i < j\). (Otherwise, swap \(x\) and \(y\).)

Let \(z = 0^{2i}1\).

Then \(xz = 10^{2i-2}10^{2i}1\) is the binary representation of \(2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2\), and therefore \(xz \in L\).

On the other hand, \(yz = 10^{2j-2}10^{2j}1\) is the binary representation of the integer \(2^{2i+2j} + 2^{2i+1} + 1\). Simple algebra gives us the inequalities

\[
(2^{i+j})^2 = 2^{2i+2j} < 2^{2i+2j} + 2^{2i+1} + 1 < 2^{2(i+j)} + 2^{i+j+1} + 1 = (2^{i+j} + 1)^2.
\]

So \(2^{2i+2j} + 2^{2i+1} + 1\) lies between two consecutive perfect squares, and thus is not a perfect square, which implies that \(yz \notin L\).

We conclude that \(F\) is a fooling set for \(L\).

Because \(F\) is infinite, \(L\) cannot be regular.