Prove that each of the following languages is not regular.

1. \( \{ \theta^{2n} \mid n \geq 0 \} \)

Solution (\( F = L \)): Let \( F = L = \{ \theta^{2n} \mid n \geq 0 \} \).

Let \( x \) and \( y \) be arbitrary distinct elements of \( F \).

Then \( x = \theta^{2i} \) and \( y = \theta^{2j} \) for some non-negative integers \( i \neq j \).

Let \( z = \theta^{2i} \).

- \( xz = \theta^{2i} \theta^{2i} = \theta^{2i+1} \in L \).
- \( yz = \theta^{2j} \theta^{2i} = \theta^{2i+2j} \not\in L \), because \( i \neq j \), and thus \( 2^i + 2^j \) is not a power of 2.

Because \( xz \in L \) and \( yz \not\in L \), the suffix \( z \) distinguishes \( x \) and \( y \).

But \( x \) and \( y \) were arbitrary, so every pair of elements in \( F \) has a distinguishing suffix.

In other words, \( F \) is a fooling set for \( L \).

We conclude that \( L \) cannot be regular, because the fooling set \( F \) is infinite.

Solution (\( F = \theta^* \)): Let \( F = \theta^* = \{ \theta^n \mid n \geq 0 \} \).

Let \( x \) and \( y \) be arbitrary distinct elements of \( F \).

Then \( x = \theta^i \) and \( y = \theta^j \) for some non-negative integers \( i \neq j \).

Without loss of generality, assume \( i < j \).

Let \( r \) be any integer such that \( 2^r > j \), and let \( z = \theta^{2^r-i} \).

Then \( xz = \theta^i \theta^{2^r-i} = \theta^{2^r} \in L \).

But \( yz = \theta^j \theta^{2^r-i} = \theta^{2^r+j-i} \not\in L \), because \( 2^r + j - i \) is not a power of 2:

\[
2^r < 2^r + j - i < 2^r + j < 2^r + 2^r = 2^{r+1}
\]

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

Solution (\( F = \theta^* \), but distinguish the other way): Let \( F = \theta^* = \{ \theta^n \mid n \geq 0 \} \).

Let \( x \) and \( y \) be arbitrary distinct elements of \( F \).

Then \( x = \theta^i \) and \( y = \theta^j \) for some non-negative integers \( i \neq j \).

Without loss of generality, assume \( i < j \).

Let \( r \) be any integer such that \( 2^{r-1} > j \), and let \( z = \theta^{2^r-j} \).

Then \( xz = \theta^i \theta^{2^r-j} = \theta^{2^r-j+i} \not\in L \), because \( 2^r - j + i \) is not a power of 2:

\[
2^{r-1} = 2^r - 2^{r-1} < 2^r - j < 2^r - j + i < 2^r
\]

But \( yz = \theta^j \theta^{2^r-j} = \theta^{2^r} \in L \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.
2. \( \{0^{2n}1^n \mid n \geq 0\} \)

**Solution \( (F = (\emptyset \emptyset)^*) \):** Let \( F \) be the language \( (\emptyset \emptyset)^* \).

Let \( x \) and \( y \) be arbitrary distinct strings in \( F \).
Then \( x = 0^{2i} \) and \( y = 0^{2j} \) for some non-negative integers \( i \neq j \).
Let \( z = 1^i \).
Then \( xz = 0^{2i}1^i \in L \).
And \( yz = 0^{2j}1^i \notin L \), because \( i \neq j \).
Thus, \( F \) is a fooling set for \( L \).
Because \( F \) is infinite, \( L \) cannot be regular.

**Solution \( (F = \emptyset^* \) \):** Let \( F \) be the language \( \emptyset^* \).
Let \( x \) and \( y \) be arbitrary distinct strings in \( F \).
Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).
Let \( z = 0^i1^i \).
Then \( xz = 0^{2i}1^i \in L \).
And \( yz = 0^{i+j}1^i \notin L \), because \( i + j \neq 2i \).
Thus, \( F \) is a fooling set for \( L \).
Because \( F \) is infinite, \( L \) cannot be regular.

**Solution (via homomorphism):** Suppose to the contrary that \( L \) is the language of some DFA \( M = (Q, s, A, \delta) \). Construct a new DFA \( M' = (Q, s, A, \delta') \) with the same states, start state, and accepting states as \( M \), but with a new transition function:

\[
\delta'(q, a) = \begin{cases} 
\delta^*(q, 0\emptyset) & \text{if } a = \emptyset \\
\delta(q, 1) & \text{if } a = 1 
\end{cases}
\]

In other words, \( M' \) simulates \( M \), but pretends that every \( \emptyset \) it reads is actually two \( \emptyset \)s.

Let \( \text{doubleoh} \) be the following string function:

\[
\text{doubleoh}(w) := \begin{cases} 
\emptyset & \text{if } w = \varepsilon \\
\emptyset \cdot \text{doubleoh}(x) & \text{if } w = \emptyset x \\
1 \cdot \text{doubleoh}(x) & \text{if } w = 1 x 
\end{cases}
\]

In particular, for any integer \( n \), we have \( \text{doubleoh}(\emptyset^n 1^n) = \emptyset^{2n}1^n \). Straightforward but tedious induction implies that our new DFA \( M' \) accepts a string \( w \) if and only if the original DFA \( M \) accepts the string \( \text{doubleoh}(w) \). It follows that \( L(M') = \{\emptyset^n 1^n \mid n \geq 0\} \).
But we proved in class that \( L(M') \) is not regular, so we have reached a contradiction; the original DFA \( M \) cannot exist!

[Yes, this proof would be worth full credit, both on homework and exams. But the fooling set argument is simpler, so try that first!]
3. \( \{0^m1^n \mid m \neq 2n\} \)

**Solution \((F = (00)^*)\):** Let \( F \) be the language \((00)^*\).

Let \( x \) and \( y \) be arbitrary distinct strings in \( F \).

Then \( x = 0^{2i} \) and \( y = 0^{2j} \) for some non-negative integers \( i \neq j \).

Let \( z = 1^i \).

Then \( xz = 0^{2i+1} \notin L \).

And \( yz = 0^{2j+1} \in L \), because \( i \neq j \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

**Solution \((F = \emptyset^*)\):** Let \( F \) be the language \( \emptyset^* \).

Let \( x \) and \( y \) be arbitrary distinct strings in \( F \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Let \( z = 0^i1^i \).

Then \( xz = 0^{2i+1} \notin L \).

And \( yz = 0^{i+j+1} \in L \), because \( i + j \neq 2i \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

**Solution (via closure properties):** If \( L \) were regular, then the language

\[
\emptyset^*1^* \setminus L = \{0^{m+1} \mid m = 2n\} = \{0^{2n+1} \mid n \geq 0\}
\]

would also be regular, because regular languages are closed under complement. But we just proved that \( \{0^{2n+1} \mid n \geq 0\} \) is not regular in problem 2.

[Yes, this proof would be worth full credit, either in homework or on an exam.]
4. Strings over \( \{0, 1\} \) where the number of 0s is exactly twice the number of 1s.

**Solution (\( F = 1^* \)):** Let \( F \) be the language \( 1^* \).

Let \( x \) and \( y \) be arbitrary distinct strings in \( F \).

Then \( x = 1^i \) and \( y = 1^j \) for some non-negative integers \( i \neq j \).

Let \( z = 0^{2i} \).

Then \( xz = 1^i0^{2i} \in L \).

And \( yz = 1^i0^{2j} \not\in L \), because \( i \neq j \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

**Solution (\( F = 0^* \)):** Let \( F \) be the language \( 0^* \).

Let \( x \) and \( y \) be arbitrary distinct strings in \( F \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Let \( z = 0^i1^i \).

Then \( xz = 0^{2i}1^i \in L \).

And \( yz = 0^{i+j}1^i \not\in L \), because \( i + j \neq 2i \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

**Solution (via closure properties):** If \( L \) were regular, then the language

\[
L \cap 0^*1^* = \{0^{2n}1^n \mid n \geq 0\}
\]

would also be regular, because regular languages are closed under intersection. But we just proved that \( \{0^{2n}1^n \mid n \geq 0\} \) is not regular in problem 2.

[Yes, this proof would be worth full credit, either in homework or on an exam.]
5. Strings of properly nested parentheses ( ), brackets [ ], and braces { }. For example, the string ([ ]){ } is in this language, but the string ([ ] ) is not, because the left and right delimiters don’t match.

**Solution:** Let $F$ be the language $\{^*\}$.  
Let $x$ and $y$ be arbitrary distinct strings in $F$.  
Then $x = (i)^i$ and $y = (j)^i$ for some non-negative integers $i \neq j$.  
Let $z = )^i$.  
Then $xz = (i)^i \in L$.  
And $yz = (j)^i \notin L$, because $i \neq j$.  
Thus, $F$ is a fooling set for $L$.  
Because $F$ is infinite, $L$ cannot be regular. 

*[Notice that this argument doesn’t even try to consider strings with different types of brackets, because it doesn’t have to. Our language $L$ has an infinite fooling set $F$; that’s enough.]*

**Solution (via closure properties):** If $L$ were regular, then the language 

$$L' := L \cap \{^*\} = \{[n]^n \mid n \geq 0\}$$

would also be regular, because regular languages are closed under intersection. But $L'$ is the same as the language $\{0^n1^n \mid n \geq 0\}$, except for renaming the symbols $0 \mapsto [ \text{ and } 1 \mapsto ]$, and we proved that $\{0^n1^n \mid n \geq 0\}$ in class.  

*[Yes, this proof would be worth full credit, either in homework or on an exam.]*
Harder problems to think about later:

6. Strings of the form \(w_1 \# w_2 \# \cdots \# w_n\) for some \(n \geq 2\), where each substring \(w_i\) is a string in \(\{0, 1\}^*\), and some pair of substrings \(w_i\) and \(w_j\) are equal.

Solution (consider the special case \(n = 2\)):

Let \(F\) be the language \(0^*\).

Let \(x\) and \(y\) be arbitrary distinct strings in \(F\).

Then \(x = 0^i\) and \(y = 0^j\) for some non-negative integers \(i \neq j\).

Let \(z = \#0^i\).

Then \(xz = 0^i \#0^i \in L\).

And \(yz = 0^i \#0^j \notin L\), because \(i \neq j\).

Thus, \(F\) is a fooling set for \(L\).

Because \(F\) is infinite, \(L\) cannot be regular. ■

[Notice that this argument doesn’t even try to consider strings with more than one \#, because it doesn’t have to. Our language \(L\) has an infinite fooling set \(F\); that’s enough.]
7. \( \{0^{n^2} \mid n \geq 0\} \)

**Solution (F = L):** Let \( x \) and \( y \) be arbitrary distinct strings in \( L \).

Without loss of generality, \( x = 0^i \) and \( y = 0^j \) for some \( i > j \geq 0 \).

Let \( z = 0^{2j+1} \).

Then \( xz = 0^{i^2+2j+1} \notin L \), because \( i^2 < i^2 + 2j + 1 < (i+1)^2 \).

On the other hand, \( yz = 0^{i^2+2i+1} = 0^{(i+1)^2} \in L \)

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( L \) is a fooling set for \( L \).

Because \( L \) is infinite, \( L \) cannot be regular. ■

**Solution (F = 0^*):** Let \( x \) and \( y \) be arbitrary distinct strings in \( 0^* \).

Without loss of generality, \( x = 0^i \) and \( y = 0^j \) for some \( i > j \geq 0 \).

Let \( z = 0^{i^2+i+1} \).

Then \( xz = 0^{i^2+2i+1} = 0^{(i+1)^2} \in L \).

On the other hand, \( yz = 0^{i^2+i+j+1} \notin L \), because \( i^2 < i^2 + i + j + 1 < (i+1)^2 \).

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( 0^* \) is a fooling set for \( L \).

Because \( 0^* \) is infinite, \( L \) cannot be regular. ■

**Solution (F = 0000^*):** Let \( x \) and \( y \) be arbitrary distinct strings in \( 0000^* \).

Without loss of generality, \( x = 0^i \) and \( y = 0^j \) for some \( i > j \geq 3 \).

Let \( z = 0^{i^2-i} \).

Then \( xz = 0^{i^2} \in L \).

On the other hand, \( yz = 0^{i^2-i+j} \notin L \), because \( (i-1)^2 = i^2 - 2i + 1 < i^2 - i < i^2 - i + j < i^2 \).

(The first inequality requires \( i \geq 2 \), and the second requires \( j \geq 1 \).)

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( 0000^* \) is a fooling set for \( L \).

Because \( 0000^* \) is infinite, \( L \) cannot be regular. ■
8. \( \{ w \in (0 + 1)^* | w \) is the binary representation of a perfect square\} 

**Solution:** We design our fooling set around numbers of the form \((2^k + 1)^2 = 2^{2k} + 2^{k+1} + 1 = 10^{k-2}10^k 1 \in L\), for any integer \( k \geq 2 \). The argument is somewhat simpler if we further restrict \( k \) to be even.

Let \( F = 1(00)^*1 \), and let \( x \) and \( y \) be arbitrary distinct strings in \( F \).

Then \( x = 10^{2i-2}1 \) and \( y = 10^{2j-2}1 \), for some positive integers \( i \neq j \).

Without loss of generality, assume \( i < j \). (Otherwise, swap \( x \) and \( y \).)

Let \( z = 0^{2i}1 \).

Then \( xz = 10^{2i-2}10^{2i}1 \) is the binary representation of \( 2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2 \), and therefore \( xz \in L \).

On the other hand, \( yz = 10^{2j-2}10^{2j}1 \) is the binary representation of the integer \( 2^{2i+2j} + 2^{2i+1} + 1 \). Simple algebra gives us the inequalities

\[
(2^{i+j})^2 = 2^{2i+2j} < 2^{2i+2j} + 2^{2i+1} + 1 \]
\[
< 2^{2(i+j)} + 2^{i+j+1} + 1 = (2^{i+j} + 1)^2.
\]

So \( 2^{2i+2j} + 2^{2i+1} + 1 \) lies between two consecutive perfect squares, and thus is not a perfect square, which implies that \( yz \notin L \).

We conclude that \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular. 

\[\blacksquare\]