1. Recall that a 5-coloring of a graph $G$ is a function that assigns each vertex of $G$ a “color” from the set \{0, 1, 2, 3, 4\}, such that for any edge $uv$, vertices $u$ and $v$ are assigned different “colors”. A 5-coloring is careful if the colors assigned to adjacent vertices are not only distinct, but differ by more than 1 (mod 5). Prove that deciding whether a given graph has a careful 5-coloring is NP-hard.

**Solution:** We prove that careful 5-coloring is NP-hard by reduction from the standard 5COLOR problem.

Given a graph $G$, we construct a new graph $H$ by replacing each edge in $G$ with a path of length three. I claim that $H$ has a careful 5-coloring if and only if $G$ has a (not necessarily careful) 5-coloring.

$\leftarrow\rightarrow$ Suppose $G$ has a 5-coloring. Consider a single edge $uv$ in $G$, and suppose $\text{color}(u) = a$ and $\text{color}(v) = b$. We color the path from $u$ to $v$ in $H$ as follows:

- If $b = (a + 1) \text{ mod } 5$, use colors $(a, (a + 2) \text{ mod } 5, (a - 1) \text{ mod } 5, b)$.
- If $b = (a - 1) \text{ mod } 5$, use colors $(a, (a - 2) \text{ mod } 5, (a + 1) \text{ mod } 5, b)$.
- Otherwise, use colors $(a, b, a, b)$.

In particular, every vertex in $G$ retains its color in $H$. The resulting 5-coloring of $H$ is careful.

$\leftarrow\rightarrow$ On the other hand, suppose $H$ has a careful 5-coloring. Consider a path $(u, x, y, v)$ in $H$ corresponding to an arbitrary edge $uv$ in $G$. There are exactly eight careful colorings of this path with $\text{color}(u) = 0$, namely: $(0, 2, 0, 2)$, $(0, 2, 0, 3)$, $(0, 2, 4, 1)$, $(0, 2, 4, 2)$, $(0, 3, 0, 3)$, $(0, 3, 0, 2)$, $(0, 3, 1, 3)$, $(0, 3, 1, 4)$. It follows immediately that $\text{color}(u) \neq \text{color}(v)$. Thus, if we color each vertex of $G$ with its color in $H$, we obtain a valid 5-coloring of $G$.

Given $G$, we can clearly construct $H$ in polynomial time. ■
2. Prove that the following problem is NP-hard: Given an undirected graph $G$, find any integer $k > 374$ such that $G$ has a proper coloring with $k$ colors but $G$ does not have a proper coloring with $k - 374$ colors.

**Solution:** Let $G'$ be the union of 374 copies of $G$, with additional edges between every vertex of each copy and every vertex in every other copy. Given $G$, we can easily build $G'$ in polynomial time by brute force. Let $\chi(G)$ and $\chi(G')$ denote the minimum number of colors in any proper coloring of $G$, and define $\chi(G')$ similarly.

$\implies$ Fix any coloring of $G$ with $\chi(G)$ colors. We can obtain a proper coloring of $G'$ with $374 \cdot \chi(G)$ colors, by using a distinct set of $\chi(G)$ colors in each copy of $G$. Thus, $\chi(G') \leq 374 \cdot \chi(G)$.

$\iff$ Now fix any coloring of $G'$ with $\chi(G')$ colors. Each copy of $G$ in $G'$ must use its own distinct set of colors, so at least one copy of $G$ uses at most $\lfloor \chi(G') / 374 \rfloor$ colors. Thus, $\chi(G) \leq \lfloor \chi(G') / 374 \rfloor$.

These two observations immediately imply that $\chi(G') = 374 \cdot \chi(G)$. It follows that if $k$ is an integer such that $k - 374 < \chi(G') \leq k$, then $\chi(G) = \chi(G') / 374 = \lfloor k / 374 \rfloor$. Thus, if we could compute such an integer $k$ in polynomial time, we could compute $\chi(G)$ in polynomial time. But computing $\chi(G)$ is NP-hard!  ■
3. A **bicoloring** of an undirected graph assigns each vertex a set of two colors. There are two types of bicoloring: In a weak bicoloring, the endpoints of each edge must use different sets of colors; however, these two sets may share one color. In a strong bicoloring, the endpoints of each edge must use distinct sets of colors; that is, they must use four colors altogether. Every strong bicoloring is also a weak bicoloring.

(a) Prove that finding the minimum number of colors in a weak bicoloring of a given graph is NP-hard.

**Solution:** It suffices to prove that deciding whether a graph has a weak bicoloring with three colors is NP-hard, using the following trivial reduction from the standard 3COLOR problem.

Let $G$ be an arbitrary undirected graph. I claim that $G$ has a proper 3-coloring if and only if $G$ has a weak bicoloring with 3 colors.

- Suppose $G$ has a proper coloring using the colors red, green, and blue. We can obtain a weak bicoloring of $G$ using only the colors cyan, magenta, and yellow by recoloring each red vertex with \{magenta, yellow\}, recoloring each blue vertex with \{magenta, cyan\}, and recoloring each green vertex with \{yellow, cyan\}.
- Suppose $G$ has a weak bicoloring using the colors cyan, magenta, yellow. Then we can obtain a proper 3-coloring of $G$ by defining red = \{magenta, yellow\}, defining blue = \{magenta, cyan\}, and defining green = \{yellow, cyan\}.

More generally, for any integer $k$ and any graph $G$, every weak $k$-bicoloring of $G$ is also a proper $\binom{k}{2}$-coloring of $G$, and vice versa. $\blacksquare$
(b) Prove that finding the minimum number of colors in a strong bicoloring of a given graph is NP-hard.

Solution: It suffices to prove that deciding whether a graph has a strong bicoloring with five colors is NP-hard, using the following reduction from the standard 3Color problem.

Let $G$ be an arbitrary undirected graph. We build a new graph from $G$ as follows:

- Add a new vertex $z$ and edges $zv$ to every vertex $v$ of $G$.
- Subdivide every edge of $G$ into a path of length $3$. (But don’t subdivide the new edges incident to $z$.)

I claim that $G$ has a proper 3-coloring if and only if $H$ has a strong bicoloring with five colors.

$\Rightarrow$ Suppose $G$ has a proper coloring with colors red, green, and blue. We obtain a strong bicoloring of $H$ with colors cyan, magenta, yellow, white, and black as follows:

- Color vertex $z$ \{white, black\}.
- Recolor the vertices of $G$ by defining red = \{magenta, yellow\} and green = \{cyan, yellow\} and blue = \{magenta, cyan\}
- Color the new vertices on each red-green edge \{cyan, black\} and \{magenta, white\}, the new vertices on each red-blue edge \{cyan, white\} and \{yellow, black\}, and the new vertices on each blue-green edge \{yellow, black\} and \{magenta, white\}.

$\Leftarrow$ On the other hand, suppose $H$ has a strong bicoloring with colors cyan, magenta, yellow, white, and black. Without loss of generality, vertex $z$ is colored \{white, black\}, and therefore each vertex of $G$ is colored either \{magenta, yellow\} or \{cyan, yellow\} or \{magenta, cyan\}.

Consider an arbitrary edge $uv$ of $G$. Suppose for the sake of argument that $u$ and $v$ are assigned the same pair of colors, without loss of generality \{magenta, yellow\}. Then the intermediate vertices on the corresponding path in $H$ only use the colors cyan, white, and black. But this is impossible, because two adjacent vertices of $H$ must use four distinct colors. Thus, $u$ and $v$ must be assigned distinct (but not disjoint!) pairs of colors.
We conclude that defining red = \{magenta, yellow\} and blue = \{magenta, cyan\} and green = \{yellow, cyan\} gives us a proper 3-coloring of \(G\).

We can easily construct \(H\) from \(G\) in polynomial time by brute force. 

Five is the smallest number of colors for which strong bicoloring is NP-hard. A graph has a strong bicoloring with four colors if and only if it is bipartite, and a strong bicoloring with two or three colors if and only if it has no edges.