Here are the formal recursive definitions of string length, concatenation, and reversal:

\[
|w| := \begin{cases} 
0 & \text{if } w = \varepsilon \\
1 + |x| & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x
\end{cases}
\]

\[
w \cdot z := \begin{cases} 
z & \text{if } w = \varepsilon \\
a \cdot (x \cdot z) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x
\end{cases}
\]

\[
w^R := \begin{cases} 
\varepsilon & \text{if } w = \varepsilon \\
x^R \cdot a & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x
\end{cases}
\]

**Lemma 1:** \(w \cdot \varepsilon = w\) for all strings \(w\).

**Lemma 2:** \(|w \cdot x| = |w| + |x|\) for all strings \(w\) and \(x\).

**Lemma 3:** \((w \cdot x) \cdot y = w \cdot (x \cdot y)\) for all strings \(w, x,\) and \(y\).

1. Prove that \(|w^R| = |w|\) for every string \(w\).

**Solution (induction on \(w\)):**

Let \(w\) be an arbitrary string.

Assume for any string \(x\) where \(|x| < |w|\) that \(|x^R| = |x|\).

There are two cases to consider.

- If \(w = \varepsilon\), then
  
  \[
  |w^R| = |\varepsilon^R| \\
  = |\varepsilon| \\
  = |w| 
  \]
  
  because \(w = \varepsilon\) by definition of \(R\) because \(w = \varepsilon\)

- Otherwise, \(w = ax\) for some symbol \(a\) and some string \(x\), and therefore
  
  \[
  |w^R| = |(ax)^R| \\
  = |x^R \cdot a| \\
  = |x^R| + |a| \\
  = |x| + 1 \\
  = |ax| \\
  = |w| 
  \]
  
  because \(w = ax\) by definition of \(R\) by Lemma 2 by definition of \(|.|\) (twice) by the induction hypothesis by definition of \(|.|\) because \(w = ax\)

In both cases, we conclude that \(|w^R| = |w|\).  ■
2. Prove that \((w \cdot z)^R = z^R \cdot w^R\) for all strings \(w\) and \(z\).

Solution (induction on \(w\)):

Let \(w\) and \(z\) be arbitrary strings.

Assume for all strings \(x\) where \(|x| < |w|\) that \((x \cdot z)^R = x^R \cdot z^R\).

There are two cases to consider:

- If \(w = \varepsilon\), then
  \[
  (w \cdot z)^R = (\varepsilon \cdot z)^R = z^R = z^R \cdot \varepsilon = z^R \cdot \varepsilon^R = z^R \cdot w^R
  \]
  because \(w = \varepsilon\) by definition of \(\cdot\)
  by Lemma 1
  by definition of \(^R\)
  because \(w = \varepsilon\)

- Otherwise, \(w = ax\) for some symbol \(a\) and some string \(x\).
  \[
  (w \cdot z)^R = (ax \cdot z)^R = (a \cdot (x \cdot z))^R = (x \cdot z)^R \cdot a = (z^R \cdot x^R) \cdot a
  \]
  by the induction hypothesis (\(\ast\))
  \[
  = z^R \cdot (x^R \cdot a) = z^R \cdot (ax)^R = z^R \cdot w^R
  \]
  by Lemma 3
  by definition of \(^R\)
  because \(w = ax\)

In both cases, we conclude that \((w \cdot z)^R = z^R \cdot w^R\).

How did I know that the induction hypothesis needs to change the first string \(w\), but not the second string \(z\)? I actually wrote down the inductive argument first, and then noticed that I needed to argue inductively about \(x \cdot z\) at line (\(\ast\)). Same string \(z\), but \(w\) changed to \(x\).

Alternatively, we could notice that the recursive definition of \(w \cdot z\) recurses on \(w\) but leaves \(z\) unchanged. Inductive proofs always mirror the recursive definitions of the objects in question.

Alternatively, in light of Lemma 2, we could have used induction on the sum of the string lengths. Then the inductive hypothesis would read “Assume for all strings \(x\) and \(y\) such that \(|x| + |y| < |w| + |z|\) that \((x \cdot y)^R = x^R \cdot y^R\).”
3. Prove that \((w^R)^R = w\) for every string \(w\).

**Solution (induction on \(w\)):**

Let \(w\) be an arbitrary string.

Assume for any string \(x\) where \(|x| < |w|\) that \((x^R)^R = x\).

There are two cases to consider.

- If \(w = \epsilon\), then
  \[
  (w^R)^R = (\epsilon^R)^R \quad \text{because } w = \epsilon
  = \epsilon^R \quad \text{by definition of } R^R
  = \epsilon \quad \text{by definition of } R
  = w \quad \text{because } w = \epsilon
  
  \]

- Otherwise, \(w = ax\) for some symbol \(a\) and some string \(x\).
  \[
  (w^R)^R = ((ax)^R)^R \quad \text{because } w = ax
  = (x^R \cdot a)^R \quad \text{by definition of } R^R
  = a^R \cdot (x^R)^R \quad \text{by problem 2}
  = a \cdot (x^R)^R \quad \text{by definition of } \cdot
  = a \cdot x \quad \text{by the induction hypothesis}
  = w \quad \text{because } w = ax
  
  \]

In both cases, we conclude that \((w^R)^R = w\).  

To think about later: Let $\#(a, w)$ denote the number of times symbol $a$ appears in string $w$. For example, $\#(X, \text{WTF374}) = 0$ and $\#(\emptyset, 00010101010100100) = 12$.

4. Give a formal recursive definition of $\#(a, w)$.

Solution:

$$
\#(a, w) = \begin{cases} 
0 & \text{if } w = \varepsilon \\
1 + \#(a, x) & \text{if } w = ax \text{ for some string } x \\
\#(a, x) & \text{if } w = bx \text{ for some symbol } b \neq a \text{ and some string } x 
\end{cases}
$$

Solution (clever notation):

$$
\#(a, w) = \begin{cases} 
0 & \text{if } w = \varepsilon \\
\#(a, x) + [a = b] & \text{if } w = bx \text{ for some symbol } b \text{ and some string } x 
\end{cases}
$$

The expression $[a = b]$ in red is Iverson bracket notation. For any proposition $P$, the expression $[P]$ is equal to 1 if $P$ is true and 0 if $P$ is false.
5. Prove that $#(a, w \cdot z) = #a(w) + #a(z)$ for all symbols $a$ and all strings $w$ and $z$.

Solution (induction on $w$):
Let $a$ be an arbitrary symbol, and let $w$ and $z$ be arbitrary strings.
Assume for any string $x$ such that $|x| < |w|$ that $#(a, x \cdot z) = #a(x) + #a(z)$.
There are three cases to consider.

- If $w = \epsilon$, then
  
  
  
  $$
  #(a, w \cdot z) = #(a, \epsilon \cdot z) \\
  = #(a, \epsilon) \\
  = #a(\epsilon) + #a(z) \\
  = #a(w) + #a(z)
  $$
  
  
  because $w = \epsilon$

- If $w = ax$ for some string $x$, then
  
  
  
  $$
  #(a, w \cdot z) = #(a, ax \cdot z) \\
  = #(a, a \cdot (x \cdot z)) \\
  = 1 + #(a, x \cdot z) \\
  = 1 + #a(x) + #a(z) \\
  = #a(ax) + #a(z) \\
  = #a(w) + #a(z)
  $$
  
  
  by definition of $\cdot$

- If $w = bx$ for some symbol $b \neq a$ and some string $x$, then
  
  
  
  $$
  #(a, w \cdot z) = #(a, bx \cdot z) \\
  = #(a, b \cdot (x \cdot z)) \\
  = #(a, x \cdot z) \\
  = #a(x) + #a(z) \\
  = #a(bx) + #a(z) \\
  = #a(w) + #a(z)
  $$
  
  
  by definition of $\cdot$

In every case, we conclude that $#(a, w \cdot z) = #a(w) + #a(z)$. ■
6. Prove that $\#(a, w^R) = \#(a, w)$ for all symbols $a$ and all strings $w$.

**Solution (induction on $w$):** Let $a$ be an arbitrary symbol, and let $w$ be an arbitrary string.

Assume for any string $x$ such that $|x| < |w|$ that $\#(a, x^R) = \#(a, x)$.

There are three cases to consider.

- If $w = \epsilon$, then $w^R = \epsilon = w$ by definition, so $\#(a, w^R) = \#(a, w)$.
- If $w = ax$ for some string $x$, then
  \[
  \#(a, w^R) = \#(a, (ax)^R) \quad \text{because } w = ax \\
  = \#(a, x^R \cdot a) \quad \text{by definition of } R \\
  = \#(a, x^R) + \#(a, a) \quad \text{by problem 5} \\
  = \#(a, x^R) + 1 \quad \text{by definition of } \# \\
  = \#(a, x^R) + 1 \quad \text{by the induction hypothesis} \\
  = \#(a, ax) \quad \text{by definition of } \# \\
  = \#(a, w) \quad \text{because } w = ax
  \]

- If $w = bx$ for some symbol $b \neq a$ and some string $x$, then
  \[
  \#(a, w^R) = \#(a, (bx)^R) \quad \text{because } w = bx \\
  = \#(a, x^R \cdot b) \quad \text{by definition of } R \\
  = \#(a, x^R) + \#(a, b) \quad \text{by problem 5} \\
  = \#(a, x^R) \quad \text{by definition of } \# \\
  = \#(a, x) \quad \text{by the induction hypothesis} \\
  = \#(a, bx) \quad \text{by definition of } \# \\
  = \#(a, w) \quad \text{because } w = ax
  \]

In every case, we conclude that $\#(a, w^R) = \#(a, w)$. ■