1. Prove that the following languages are *not* regular.

(a) $\{0^m 1^n \mid m \text{ and } n \text{ are relatively prime}\}$

Solution: Consider the set $F = \{0^p \mid p \text{ is prime}\}$.
Let x and y be arbitrary distinct strings in F.
Then $x = 0^i$ and $y = 0^j$ for some prime integers $i \neq j$.
We will set $z = 1^i$. Note that i will be relatively prime with j but not with i.

- Then $xz = 0^i 1^i \notin L$, because i is not relatively prime with i.
- But $yz = 0^j 1^i \in L$, because j and i are relatively prime.

Thus, z is a distinguishing suffix for x and y.
We conclude that F is a fooling set for L.
Because F is infinite (since there are infinitely many primes), L cannot be regular. ■

Rubric: 3 points: standard fooling set rubric (scaled). This is not the only correct solution.

(b) $\{w \in (0+1)^* \mid 10^n 1^n \text{ for } n > 0 \text{ is a suffix of } w\}$

Solution: Consider the set $F = \{10^n | n > 0\}$.
Let x and y be arbitrary distinct strings in F.
Then $x = 10^i$ and $y = 10^j$ for some integers $i \neq j$, $i, j \geq 1$.
Let $z = 1^j$.

- $xz = 10^i 1^j \notin L$, because $i \neq j$.
- $yz = 10^j 1^j \in L$.

Thus, z is a distinguishing suffix for x and y.
We conclude that F is a fooling set for L.
Because F is infinite, L cannot be regular. ■

Rubric: 3 points: standard fooling set rubric (scaled). This is not the only correct solution.

- We are really reasoning about the language $L \cap \{10^n 1^n | n > 0\}$.
- We really do need $j \geq 1$, because $yz \notin L$ when $j = 0$.

(c) The set of all palindromes in $(0+1)^*$ whose length is divisible by 3.

Solution: Consider the set $F = (000)^*111 = \{0^{3n}1^3 \mid n \geq 0\}$.
Let x and y be arbitrary distinct strings in F.
Then $x = 0^{3i}1^3$ and $y = 0^{3j}1^3$ for some integers $i \neq j$.
Let $z = 0^{3i}$.

- Then $xz = 0^{3i}1^30^{3i}$ is a palindrome of length $3(2i+1)$, so $xz \in L$.
- But $yz = 0^{3j}1^30^{3i}$ is not a palindrome (because $i \neq j$), so $yz \notin L$.

Thus, z is a distinguishing suffix for x and y.
We conclude that F is a fooling set for L.
Because F is infinite, L cannot be regular. ∎

Solution: Consider the set $F = (000)^*0 = \{0^{3n+1} \mid n \geq 0\}$.
Let x and y be arbitrary distinct strings in F.
Then $x = 0^{3i+1}$ and $y = 0^{3j+1}$ for some integers $i \neq j$.
Let $z = 10^{3i+1}$.

- Then $xz = 0^{3i+1}10^{3i+1}$ is a palindrome of length $2(3i+1)+1 = 3(2i+1)$, so $xz \in L$.
- But $yz = 0^{3j+1}10^{3i+1}$ is not a palindrome (because $i \neq j$), so $yz \notin L$.

Thus, z is a distinguishing suffix for x and y.
We conclude that F is a fooling set for L.
Because F is infinite, L cannot be regular. ∎

Rubric: 4 points: standard fooling set rubric (scaled). These are not the only correct solutions.

2. For each of the following languages over the alphabet $\Sigma = \{0, 1\}$, either prove that the language is regular (by constructing an appropriate DFA, NFA, or regular expression) or prove that the language is not regular (by constructing an infinite fooling set).

   (a) $\{0^a 1 w 1 0^c \mid w \in \Sigma^*, (a \le |w| + c) \text{ and } (|w| \le a + c \text{ or } c \le a + |w|)\}$

Solution (fooling set): Not regular. Consider the set $F = 0^*$.

Let x and y be arbitrary distinct strings in F.

Then $x = 0^i$ and $y = 0^j$ for some integers $i \ne j$, $i, j \ge 0$.

Without loss of generality, assume $i < j$.

Let $z = 1^{i+2}$.

- Then $xz = 0^i 1(1^i)1 = 0^a 1(1^b)1 0^c$ where $(a, b, c) = (i, i, 0)$.
  We have $a = i \le i = b + c$ and $b = i \le i = a + c$, which implies $xz \in L$.
- And $yz = 0^j 1 1^i 1 = 0^a 1^b 0^c$ where $(a, b, c) = (j, i, 0)$.
  We have $a = j > i = b + c$, which implies $yz \notin L$.

Thus, z is a distinguishing suffix for x and y.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Rubric: 2½ points: standard fooling set rubric (scaled). This is not the only correct fooling set argument. We really do need to assume $i < j$; otherwise, $yz \in L$.

The main idea of this solution is to impose additional structure that forces $c = 0$. We are really reasoning about the language $L \cap 0^* 1^* = \{0^a 1^b \mid a \le b\}$.

(b) $\left\{ 0^a w 0^a \;\middle|\; w \in \Sigma^+, a > 0, |w| \geq 0 \right\}$

Solution: **Regular.** This is the language $\mathbf{0(0+1)^+0}$.

- Consider an arbitrary string $u \in L$.
  By definition of L, we have $u = 0^a w 0^a$ for some integer $a > 0$ and string $w \in (0+1)^+$. Since $0^a = 00^{a-1} = 0^{a-1}0$, we have that $u = 0(0^{a-1} w 0^{a-1})0$ where $(a-1) \geq 0$ and $w \in \Sigma^+$. It follows that $u \in 0(0+1)^+0$.
- On the other hand, consider an arbitrary string $u \in 0(0+1)^+0$.
  We immediately have $u = 0^a w 0^a$ for integer $a = 1$ and string $w \in (0+1)^+$.
  We conclude that $u \in L$.

■

Rubric: 2½ points: ½ for "regular" + 1 for regular expression + 1 for justification (= ½ for "if" + ½ for "only if"). This is more detail than necessary for full credit. This is not the only correct solution.

(c) $\{xww^Ry \mid w, x, y \in \Sigma^+\}$

Solution: Regular. This is the language $(0+1)^+(00+11)(0+1)^+$ of all strings of length at least 4 that have the substring 00 or 11 preceded and succeeded by at least one symbol.

- Let z be an arbitrary string in L.
  By definition, $z = xww^Ry$ for some non-empty strings x, w and y.
  Because $w \neq \varepsilon$, we have $w = p \bullet a$ for some string p and symbol a.
  The definition of reversal implies $w^R = ap^R$.
  Thus, $z = xpaap^Ry$ contains substring aa.
  The remaining substrings xp and p^Ry contain at least one symbol each, because x and y are nonempty.
  We conclude that $z \in (0+1)^+(00+11)(0+1)^+$.
- On the other hand, let z be an arbitrary string in $(0+1)^+(00+11)(0+1)^+$.
  Then $z = xaay$ for some symbol a and some nonempty strings x and y.
  Because $a = a^R$, we have $z = xaa^Ry$, which implies $z \in L$.

■

Rubric: 2½ points: ½ for "regular" + 1 for regular expression + 1 for justification (= ½ for "if" + ½ for "only if"). This is more detail than necessary for full credit.

(d) $\{ww^Rxy \mid w,x,y \in \Sigma^+\}$

Solution: Not regular. Consider the set $F = 1(00)^*01$.

Let u and v be arbitrary distinct strings in F.

Then $u = 10^{2i+1}1$ and $v = 10^{2j+1}1$ for some non-negative integers $i \neq j$.

Let $z = 10^{2i+1}111$.

- Then $uz = 10^{2i+1}110^{2i+1}111 = ww^Rxy$, where $w = u$ and $x, y = 1$, so $uz \in L$.
- For the sake of argument, suppose $vz = 10^{2j+1}110^{2i+1}111 \in L$.
  Then $vz = ww^Rxy$ for some non-empty strings w, x and y.
  The first two symbols of vz are different, so $|w| > 1$.
  The prefix w begins with 10 (the first two symbols of vz).
  So its reversal w^R must end with 01.
  The substring 01 appears exactly twice in vz.
  So there are only two possibilities for the substring ww^R.
  - $ww^R = 10^{2j+1}1$ is impossible because $|ww^R|$ must be even.
  - $ww^R = 10^{2j+1}110^{2i+1}1$ is impossible because ww^R must be a palindrome, and $i \neq j$.

  We have derived a contradiction, which implies that $vz \notin L$.

Thus, z is a distinguishing suffix for u and v.

We conclude that F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Rubric: 2½ points: standard fooling set rubric (scaled). This is not the only correct solution.

The main idea here is to impose additional structure that forces the prefix w to be arbitrarily long. We are really reasoning about the language

$$L \cap 1(00)^*011(00)^*0111 = \{10^n110^n111 \mid n \text{ is odd}\}.$$