1. Consider the following pair of mutually recursive functions on strings:

$$odds(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ a \cdot evens(x) & \text{if } w = ax \end{cases} \qquad evens(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ odds(x) & \text{if } w = ax \end{cases}$$

(a) Give a self-contained recursive definition for the function *evens* that does not involve the function *odds*.

Solution:

$$evens(w) = \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ \varepsilon & \text{if } w = a \text{ for some symbol } a \in \Sigma \\ b \cdot evens(x) & \text{if } w = abx \text{ for some } a, b \in \Sigma \text{ and some } x \in \Sigma^* \end{cases}$$

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Rubric: 2 points = ½ for case breakdown + ½ for base cases + 1 for final recursive case.

(b) Prove the following identity for all strings w and x:

$$evens(w \bullet x) = \begin{cases} evens(w) \bullet evens(x) & \text{if } |w| \text{ is even,} \\ evens(w) \bullet odds(x) & \text{if } |w| \text{ is odd.} \end{cases}$$

Solution: Let w and x be arbitrary strings.

Assume for all strings y shorter than w that

$$evens(y \bullet x) = \begin{cases} evens(y) \bullet evens(x) & \text{if } |y| \text{ is even,} \\ evens(y) \bullet odds(x) & \text{if } |y| \text{ is odd.} \end{cases}$$

There are **three** cases to consider, mirroring our recursive definition in part (a).

- Suppose $w = \varepsilon$. Then we have

$$\begin{aligned} evens(w \bullet x) &= evens(\varepsilon \bullet x) && \text{because } w = \varepsilon \\ &= evens(x) && \text{by definition of } \bullet \\ &= \varepsilon \bullet evens(x) && \text{by definition of } \bullet \\ &= evens(\varepsilon) \bullet evens(x) && \text{by definition of } evens \\ &= evens(w) \bullet evens(x) && \text{because } w = \varepsilon \end{aligned}$$

  Finally, $|w| = 0$ is even, by definition of length.

- Suppose $w = a$ for some symbol $a \in \Sigma$. Then we have

$$\begin{aligned} evens(w \bullet x) &= evens(a \bullet x) && \text{because } w = a \\ &= odds(x) && \text{by definition of } evens \\ &= \varepsilon \bullet odds(x) && \text{by definition of } \bullet \\ &= evens(a) \bullet odds(x) && \text{by definition of } evens \\ &= evens(w) \bullet odds(x) && \text{because } w = \varepsilon \end{aligned}$$

  Finally, $|w| = 1$ is odd, by definition of length.

- Suppose $w = aby$ for some symbols $a, b \in \Sigma$ and string $y \in \Sigma^*$.[a]
  In this case, there are two subcases to consider, mirroring the cases in the statement we are trying to prove.

  - Suppose $|y|$ is even. Then we have

$$\begin{aligned} evens(w \bullet x) &= evens(aby \bullet x) && \text{because } w = aby \\ &= b \cdot evens(y \cdot x) && \text{by definition of } evens \\ &= b \cdot (evens(y) \bullet evens(x)) && \text{by the induction hypothesis} \\ &= (b \cdot evens(y)) \bullet evens(x)) && \text{because } \bullet \text{ is associative} \\ &= odds(by) \bullet evens(x) && \text{by definition of } evens \\ &= evens(aby) \bullet evens(x) && \text{by definition of } evens \\ &= evens(w) \bullet evens(x) && \text{because } w = aby \end{aligned}$$

Finally, $|w| = |aby| = |y| + 2$ is even, by definition of length.

– Suppose $|y|$ is odd. Then we have

$$\begin{aligned}
evens(w \bullet x) &= evens(aby \bullet x) && \text{because } w = aby \\
&= b \cdot evens(y \cdot x) && \text{by definition of } evens \\
&= b \cdot (evens(y) \bullet odds(x)) && \text{by the induction hypothesis} \\
&= (b \cdot evens(y)) \bullet odds(x)) && \text{because } \bullet \text{ is associative} \\
&= odds(by) \bullet odds(x) && \text{by definition of } evens \\
&= evens(aby) \bullet odds(x) && \text{by definition of } evens \\
&= evens(w) \bullet odds(x) && \text{because } w = aby
\end{aligned}$$

Finally, $|w| = |aby| = |y| + 2$ is odd, by definition of length.

In every case, we conclude that

$$evens(w \bullet x) = \begin{cases} evens(w) \bullet evens(x) & \text{if } |w| \text{ is even,} \\ evens(w) \bullet odds(x) & \text{if } |w| \text{ is odd.} \end{cases}$$

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[a] We can't use the variable x here because it's already in use.

Rubric: 8 points, standard induction rubric (scaled). **This is not the only correct proof.**

Every correct proof must consider four distinct cases, to capture the case distinctions in the recursive definition of strings, in the recursive definition of *evens*, and in the statement that we are proving:

- w is empty
- w has length 1
- w has positive even length
- w has odd length greater than 1

These cases could be clustered into coarser cases with subcases (as above), or presented as four flat cases. The even/odd cases could be based on the length of w (implying similar conditions on $|y|$), or based on the length of y (implying similar conditions on $|w|$.) But no matter how the proof is organized, the case structure should be clearly *without* reading the rest of the proof.

Finally, it is not necessary to use the standalone definition of *evens* from part (a). The lines in gray show how to apply the original definitions of *evens* and *odds*.

2. Consider the following recursive function that perfectly shuffles two strings together:

$$shuffle(w,z) := \begin{cases} z & \text{if } w = \varepsilon \\ a \cdot shuffle(z,x) & \text{if } w = ax \end{cases}$$

(a) Prove that $shuffle(odds(w), evens(w)) = w$ for every string w.

Solution: Let w be an arbitrary string.
Assume for every string x shorter than w that $shuffle(odds(x), evens(x)) = x$.
There are two cases to consider, mirroring the definition of *shuffle*:

- Suppose $w = \varepsilon$.

$$\begin{aligned} & shuffle(odds(w), evens(w)) \\ &\quad = shuffle(odds(\varepsilon), evens(\varepsilon)) && \text{because } w = \varepsilon \\ &\quad = shuffle(\varepsilon, evens(\varepsilon)) && \text{by definition of } odds \\ &\quad = shuffle(\varepsilon, \varepsilon) && \text{by definition of } evens \\ &\quad = \varepsilon && \text{by definition of } shuffle \\ &\quad = w && \text{because } w = \varepsilon \end{aligned}$$

- Suppose $w = ax$ for some symbol a and some string x.

$$\begin{aligned} & shuffle(odds(w), evens(w)) \\ &\quad = shuffle(odds(ax),\ evens(ax)) && \text{because } w = ax \\ &\quad = shuffle(a \cdot evens(x),\ evens(ax)) && \text{by definition of } odds \\ &\quad = shuffle(a \cdot evens(x),\ odds(x)) && \text{by definition of } evens \\ &\quad = a \cdot shuffle(odds(x),\ evens(x)) && \text{by definition of } shuffle \\ &\quad = ax && \text{by the induction hypothesis} \\ &\quad = w && \text{because } w = ax \end{aligned}$$

In both cases, we conclude that $shuffle(odds(w), evens(w)) = w$. ■

Rubric: 5 points, standard induction rubric (scaled)

(b) Prove $evens(shuffle(w,z)) = z$ for all strings w and z such that $|w| = |z|$.

Solution: Let w and z be arbitrary strings such that $|w| = |z|$.
Assume that $evens(shuffle(w',z')) = z'$ for all strings w' and z' such that $|w'| = |z'| < |w|$.
There are two cases to consider.

- Suppose $|w| = |z| = 0$. Then $w = \varepsilon$ and $z = \varepsilon$ by definition of length.

$$\begin{aligned}
&evens(shuffle(w,z)) \\
&\quad = evens(shuffle(\varepsilon,\varepsilon)) && \text{because } w = \varepsilon \text{ and } z = \varepsilon \\
&\quad = evens(\varepsilon) && \text{by definition of } shuffle \\
&\quad = \varepsilon && \text{by definition of } evens \\
&\quad = z && \text{because } z = \varepsilon
\end{aligned}$$

- Suppose $|w| = |z| > 0$.

  The definition of length implies that $w = ax$ and $z = by$ for some symbols a and b and some strings x and y.

  The definition of length also implies that $|w| = 1 + |x|$ and $|z| = 1 + |y|$, and therefore $|x| = |y|$.

$$\begin{aligned}
&evens(shuffle(w,z)) \\
&\quad = evens(shuffle(ax,by)) && \text{because } w = ax \text{ and } z = by \\
&\quad = evens(a \cdot shuffle(by,x)) && \text{by definition of } shuffle \\
&\quad = odds(shuffle(by,x)) && \text{by definition of } evens \\
&\quad = odds(b \cdot shuffle(x,y)) && \text{by definition of } shuffle \\
&\quad = b \cdot evens(shuffle(x,y)) && \text{by definition of } odds \\
&\quad = b \cdot y && \text{by the inductive hypothesis!!} \\
&\quad = z && \text{because } z = by
\end{aligned}$$

In both cases, we conclude that $evens(shuffle(w,z)) = z$. ■

Rubric: 5 points, standard induction rubric (scaled). This is not the only correct proof!