For each statement below, check “Yes” if the statement is ALWAYS true and “No” otherwise, and give a brief explanation of your answer.

(a) Every integer in the empty set is prime.

Yes  No  vacuous / empty set contains no integers

(b) The language \( \{0^m1^n \mid m + n \leq 374 \} \) is regular.

Yes  No  \[
0^11^3\cup0^11^7\cup\cdots\cup0^3^3\cup0^3^1\cup0^1\]

(c) The language \( \{0^m1^n \mid m - n \leq 374 \} \) is regular.

Yes  No  Fooling set : 0*

(d) For all languages \( L \), the language \( L^* \) is regular.

Yes  No  \[
\forall 0^1\mid n \geq 0\]

(e) For all languages \( L \), the language \( L^* \) is infinite.

Yes  No  NOT TRUE when \( L = \{\varepsilon\} \Rightarrow L^* = \{\varepsilon, \varepsilon^2, \ldots\} \).  (Another: \( L = \emptyset \))

(f) For all languages \( L \subset \Sigma^* \), if \( L \) can be represented by a regular expression, then \( \Sigma^* \setminus L \) is recognized by a DFA.

Yes  No  If \( L \) is regular, flip accepting and rejecting states in DFA for \( L \).

(g) For all languages \( L \) and \( L' \), if \( L \cap L' = \emptyset \) and \( L' \) is not regular, then \( L \) is regular.

Yes  No  0^1^n is not regular, 1^n0 is not regular \( \forall n \geq 1 \).

(h) Every regular language is recognized by a DFA with exactly one accepting state.

Yes  No  No \( \varepsilon \)- Transitions

(i) Every regular language is recognized by an NFA with exactly one accepting state.

Yes  No  \( \varepsilon \)- transition from all acc states to a single acc state.

(j) Every language is either regular or context-free.

Yes  No  0^1^2^n
For each of the following languages over the alphabet $\Sigma = \{0, 1\}$, either prove that the language is regular or prove that the language is not regular. **Exactly one of these two languages is regular.** Both of these languages contain the string 001010000110100.

1. $\{0^n w 0^n \mid w \in \Sigma^+ \text{ and } n > 0\}$

   Let $z \in \{0^n w 0^n \mid w \in \Sigma^+, n > 0\}$.
   
   Then $z = \underbrace{0 \ldots 0}_n w \underbrace{0 \ldots 0}_n = 0 \underbrace{000 \ldots 0}_n w 000 \ldots 0 \in 0(0+1)^+ 0$

   Let $z \in 0(0+1)^+ 0$.
   
   Then $z = 0^i w 0^j$ where $w \in \Sigma^+, i \neq j, i, j > 0$.

2. $\{w^n w \mid w \in \Sigma^+ \text{ and } n > 0\}$

   $00^{n+1}0, \ 101, \ 101^2, \ 1^3 01^2, \ 1^4 01^3, \ldots$

   $F = \{1^n 0^n \mid n > 0\}$

   Let $x, y$ be any strings in $F$.

   $x = 1^i 0, \ y = 1^j 0, \ i \neq j, \ i, j > 0$.

   $z = 1^i$

   $xz = 1^i 01^i \in L, \ yz = \underbrace{1^i 01^i}_{w_1} \underbrace{1^j 01^j}_{w_2} \in L, \ w_1 + w_2 = yz \notin L$.

   $F$ is infinite foding set. So, $L$ is not regular.
The parity of a bit-string $w$ is 0 if $w$ has an even number of 1s, and 1 if $w$ has an odd number of 1s. For example:

$$\text{parity}(e) = 0 \quad \text{parity}(0010100) = 0 \quad \text{parity}(00101110100) = 1$$

(a) Give a self-contained, formal, recursive definition of the parity function. (In particular, do not refer to # or other functions defined in class.)

$$\text{parity}(w) = \begin{cases} 
0 & \text{if } w = e \\
\text{parity}(x) & \text{if } w = 0 \cdot x \\
1 & \text{parity}(x) & \text{if } w = 1 \cdot x 
\end{cases}$$

(b) Let $L$ be an arbitrary regular language. Prove that the language $\text{OddParity}(L) := \{w \in L \mid \text{parity}(w) = 1\}$ is also regular.

Product construction of $M$ and $M'$, strings in $L$ that have odd number of 1s.

$M = (\Sigma, S, \delta, s_0, F)$ is the DFA for $L$, (exists because $L$ is regular)

$M' = (\Sigma, S, \delta', s'_0, F')$ is the DFA that on input $w$ computes $\text{parity}(w)$.

Accepting states = $\delta(a, a')$ s.t.

(in product) $a \in A$, $a' \in A'$, $F$

(c) Let $L$ be an arbitrary regular language. Prove that the language $\text{AddParity}(L) := \{\text{parity}(w) \cdot w \mid w \in L\}$ is also regular.

$\text{OddParity}(L)$ is regular.

Similarly $\text{EvenParity}(L)$ is regular. [Change $M'$ to flip acc, reject states]

$$\text{AddParity}(L) = 0 \cdot \text{EvenParity}(L) + 1 \cdot \text{OddParity}(L)$$

[Hint: Yes, you have enough room.]
For each of the following languages $L$, give a regular expression that represents $L$ and describe a DFA that recognizes $L$. You do not need to prove that your answers are correct.

(a) All strings in $(0 + 1)^*$ that do not contain the substring $0110$.

\[
1^* \left(0 \left(3^* + 1 + 1111^*\right)\right)^* 1^* \]

(b) All strings in $0^*10^*$ whose length is a multiple of 3.

Product construction with Acc states, $A \times A'$. 

\[
(000)^*010(000)^* + (000)^*100(000)^* + (000)^*001(000)^*
\]
For any string $w \in \{0,1\}^*$, let $\text{oblivate}(w)$ denote the string obtained from $w$ by removing every 1. For example:

\[
\begin{align*}
\text{oblivate}(\varepsilon) &= \varepsilon \\
\text{oblivate}(000000) &= 000000 \\
\text{oblivate}(111111) &= \varepsilon \\
\text{oblivate}(01001101) &= 00000 
\end{align*}
\]

Let $L$ be an arbitrary regular language.

1. **Prove** that the language $\text{OBLIVIATE}(L) = \{\text{oblivate}(w) \mid w \in L\}$ is regular.

\[
x \in \text{OBL}(L) \iff x = \text{oblivate}(w), w \in L.
\]

Let $M$ be DFA for $L$, $M = (Q, \delta, \epsilon, s, F)$.

Let $M'$ be NFA for $\text{OBL}(L)$, $M' = (Q', \delta', \epsilon', s', F')$.

$s' = s$

$A' = A$

$s'(q, \varepsilon) = \delta(q, \varepsilon)$

Changing $\rightarrow$ arrows in $M$ to $\varepsilon$ arrows in $M'$.

2. **Prove** that the language $\text{UNOBLIVIATE}(L) = \{w \in \{0,1\}^* \mid \text{oblivate}(w) \in L\}$ is regular.

\[
w \in \text{UNOBL}(L) \iff \text{oblivate}(w) \in L.
\]

Let $M$ be a DFA for $L$, $M = (Q, \delta, \epsilon, s, F)$.

Let $M'$ be NFA for $\text{OBLV}(L)$, $M' = (Q, \delta', \epsilon', s', A', F')$.

$s' = s$

$s'(q, 0) = \delta(q, 0)$

$i.e. \text{remove } 1$s

Changing $\rightarrow$ arrows to $0 \rightarrow 0$. 

\[
\begin{align*}
\delta(q, 0) &= 0 \\
\delta(q, 1) &= q
\end{align*}
\]